Samuel Noorway Jr

THE

ELEMENTS

OF THE

DIFFERENTIAL CALCULUS;

COMPREHENDING THE

GENERAL THEORY OF CURVE SURFACES,

AND OF

CURVES OF DOUBLE CURVATURE.

INTENDED FOR THE USE OF

MATHEMATICAL STUDENTS IN SCHOOLS AND UNIVERSITIES.

BY J. R. YOUNG,

AUTHOR OF

"THE ELEMENTS OF ANALYTICAL GEOMETRY."

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ADVERTISEMENT.

THIS edition of YOUNG'S Differential and Integral Calculus is presented to the American public, with a confidence in its favourable reception, proportionate to that which the original acquired in England. The text has not been materially altered, though many errors have been corrected, some of which by Professor DODD of Princeton College, N. J.

These volumes will be found to contain a full elementary course of the subject of which they treat, and well adapted as a text book for Colleges and Universities.

The second volume, treating exclusively of the Integral Calculus, is now in press, and will be speedily published.

NEW-YORK, March, 1833.

PREFACE.

THE object of the present volume is to teach the principles of the *Differential Calculus*, and to show the application of these principles to several interesting and important inquiries, more particularly to the general theory of Curves and Surfaces. Throughout these applications I have endeavoured to preserve the strictest rigour in the various processes employed, so that the student who may have hitherto been accustomed only to the pure reasoning of the ancient geometry will not, I think, find in these higher order of researches any principle adopted, or any assumption made, inconsistent with his previous notions of mathematical accuracy. If I have, indeed, succeeded in accomplishing this very desirable object, and have really shown that the applications of the Calculus do not necessarily involve any principle that will not bear the most scrupulous examination, I may, perhaps, be allowed to think that I have, in this small volume, contributed a little towards the perfecting of the most powerful instrument which the modern analysis places in the hand of the mathematician.

It is the adoption of exceptionable principles, and even, in some cases, of contradictory theories, into the elements of this science, that have no doubt been the chief causes why it has hitherto been so little studied in a country where the ancient geometry has been so extensively and so successfully cultivated. The student who proceeds from the works of *Euclid* or of *Apollonius* to study those of our modern analysts, will be naturally enough startled to find that in the *theory* of the Differential Calculus he is to consider that as absolutely *nothing* which, in the *application* of that theory, is to be considered a quantity *infinitely small.* He will naturally enough be startled to find that a conclusion is to be taken as general, when he is at the

same time told that the process which led to that conclusion has failing cases; and yet one or both of these inconsistencies pervade more or less every book on the Calculus which I have had an opportunity of examining.

The whole theory of what the French mathematicians vaguely call *consecutive points* and *consecutive elements*, involves the first of these objectionable principles;* for, if the abscissa of any point be represented by x, then the abscissa of the consecutive point, or that separated from the former by an infinitely small interval, is represented by $x + dx$, although dx, at the outset of the subject, is said to be 0. Again, the theory of tangents, the radius of curvature, principles of osculation, &c., are all made to depend upon Taylor's theorem, and therefore can strictly apply only at those points of the curve where this theorem does not fail: the conclusions, however, are to be received in all their generality.†

* It is to be regretted that terms so vague and indefinite should be introduced into the *exact sciences;* and it is more to be regretted that English elementary writers should adopt them merely because they are used by the French, and that too without examining into the import these terms carry in the works from which they are copied. In a recent production of the University of Cambridge, the author, in attempting to follow the French mode of solving a certain problem, has confounded *consecutive points* with *consecutive elements*, two very distinct things: although neither very intelligible, the consequence of this mistake is, that the result is not what was intended; so that, after the process is fairly finished, a new counterbalancing error is introduced, and thus the solution righted!

† I am anxious not to be misunderstood here, and shall therefore state specifically the nature of my objection. In establishing the theory of contact, &c., by aid of Taylor's theorem, it is assumed that a value may be given to the increment h so small as to render the term into which it enters greater than all the following terms of the series taken together. Now how can a function of absolutely indeterminate quantities be shown to be greater or less than a series of other functions of the same indeterminate quantities without, at least, assuming some determinate relation among them? If we say that the assertion applies, whatever particular value we substitute for the indeterminate in the proposed functions or differential coefficients, we merely shift the dilemma, for an indefinite number of these particular values may render the functions all infinite; and we shall be equally at a loss to conceive how one of these infinite quantities can be greater or less than the others. It appears, therefore, that the usual process by which the theory of contact is established, applies rigorously only to those points of curves for which Taylor's development does not fail, and I cannot help thinking that on these grounds the *Analytical Theory of Functions, by Lagrange*, in its application to Ge-

If this statement be true, it is not to be wondered at that students so often abandon the study of this science, less discouraged with its difficulties than disgusted with its inconsistencies. To remove these inconsistencies, which so often harass and impede the student's progress, has been my object in the present volume; and, although my endeavours may not have entirely succeeded, I have still reason to hope that they have not entirely failed. The following brief outline will convey a notion of the extent and pretensions of the book; a more detailed enumeration of the various topics treated of, will be found in the table of contents.

I have taken for the basis of the theory the method of *limits* first employed by *Newton*, although designated by foreign writers as the *method of d'Alembert.* I consider this method to be as unexceptionable as that of *Lagrange*, and on account of its greater simplicity, better adapted to elementary instruction.

The First Chapter is devoted to the exposition of the fundamental principles; and in explaining the notation I have been careful to impress upon the student's mind that the differentials dx, dy, &c. are in themselves absolutely of no value, and that their ratios only are significant: this is the foundation of the whole theory, and it has been adhered to throughout the volume, without any shifting of the hypothesis.

In the Second Chapter it is shown, that if fx represent any function of x, and x be changed into $x + h$, the new state $f(x + h)$ of the function may always be developed according to the ascending integral powers of the increment h; and this leads to the important conclusion that the coefficient of the second term in the development of the function $f(x + h)$ is the differential coefficient derived from the function fx; a fact which *Lagrange* has made the foundation of his

ometry is defective, although I feel anxious to express my opinion of that celebrated performance with all becoming caution and humility. Indeed Lagrange himself has admitted this defect, and observes, (*Théorie des Fonctions, p.* 181,) "Quoique ces exceptions ne portent aucune atteinte à la théorie générale, il est nécessaire, pour ne rien laisser à desirer, de voir comment elle doit être modifier dans les cas particuliers dont il s'agit." (*See note C at the end.*) But he has not modified the expression deduced from this exceptionable theory for the radius of curvature, which indeed is always applicable whether the differential coefficients become infinite or not, although, for reasons already assigned, the process which led to it restricts its application to particular points.

theory of analytical functions. The chapter then goes on to treat of the differentiation of the various kinds of functions, algebraic and transcendental, direct and inverse, and concludes with an article on successive differentiation.

The Third Chapter is devoted to *Maclaurin's theorem*, and its application is shown in the development of a great variety of functions. Occasion is taken, in the course of this chapter, to introduce to the student's attention some valuable analytical formulas and expressions from *Euler*, *Demoivre*, *Cotes*, and other celebrated analysts, together with those curious properties of the circle discovered by *Cotes* and *Demoivre*.

The Fourth Chapter is on *Taylor's theorem*, which makes known the actual development of the function $f(x + h)$ according to the form established in the second chapter. From this theorem are derived commodious expressions for the total differential coefficient when the function is complicated, and whether its form be explicit or implicit; the whole being illustrated by a variety of examples.

The Fifth Chapter contains the complete theory of vanishing fractions.

The Sixth is on the maxima and minima values of functions of a single variable, and will, I think, be found to contain several original remarks and improved processes.

Chapter the Seventh is on the differentiation and development of functions of two independent variables. The usual method of obtaining the development of a function of two variables according to the powers of the increments, is to develop first on the supposition that x only varies and that y is constant, and afterwards to consider y, which is *assumed* to enter into the coefficients, to be changed into $y + h$. But y may be so combined with x in the function $\mathrm{F}(x, y)$ that it shall, when considered as a constant, disappear from all the differential coefficients, which circumstances should be pointed out and be shown not to affect the truth of the result: I have, however, avoided the necessity of showing this, by proceeding rather differently. The chapter concludes with *Lagrange's Theorem*, concisely demonstrated and applied to several examples.

The Eighth Chapter completes the theory of maxima and minima, by applying the principles delivered in chapter VI. to functions of two independent variables, and it also contains an important article on

changing the independent variable, a subject very improperly omitted in all the English books.

The Ninth Chapter is devoted to a matter of considerable importance, viz. to the examination of the cases in which Taylor's theorem fails; and I have, I think, satisfactorily shown, that these failing cases are always indicated by the differential coefficients becoming infinite, and that the theorem does not fail when these coefficients become imaginary, as *Lacroix*, and others after him, have asserted. Besides the correction of this erroneous doctrine, which has been sanctioned by names of the highest reputation, another very remarkable oversight, though of far less importance, is detected in the *Calcul des Fonctions* of *Lagrange*, and is pointed out in the present chapter: it has been unsuspectingly copied by other writers; and thus an entirely wrong solution to a very simple problem has been printed, and reprinted, without any examination into the principles employed in it; and which, I suppose, the high reputation of Lagrange was considered to render unnecessary.

These nine chapters constitute the First Section of the work, and comprise the pure theory of the subject; the remaining part is devoted to the application of this to geometry, and is divided into two parts, the first containing the theory of plane curves, and the second the theory of curve surfaces, and of curves of double curvature.

The First Chapter in the Second Section explains the method of tangents, and the general differential equation of the tangent to any plane curve is obtained by the same means that the equation is obtained in analytical geometry, and is therefore independent of the failing cases of Taylor's theorem. The method of tangents naturally leads to the consideration of rectilinear asymptotes, which is, therefore, treated of in this chapter, and several examples are given, as well when the curve is referred to polar as to rectangular coordinates, and a few passing observations made on the circular asymptotes to spiral curves, the chapter terminating with the differential expression for the arc of any plane curve determined without the aid of Taylor's theorem.

The Second Chapter contains the theory of osculation, which is shown to be unaffected by the failing cases of Taylor's theorem, although this is employed to establish the theory. The expressions for the radius of curvature are afterwards deduced, and several examples

of their application given principally to the curves of the second order, and an instance of their utility shown in determining the ratio of the earth's diameters.

The Third Chapter is on involutes, evolutes, and consecutive curves, and contains some interesting theorems and practical examples. Of what the French call consecutive curves, I have endeavoured to give a clear and satisfactory explanation, unmixed with any vague notions about infinity.

The Fourth Chapter is on the singular points of curves, and contains easy rules for detecting them, from an examination of the equation of the curve. This chapter also contains the general theory of curvilinear asymptotes, and completes the Second Section, or that assigned to the consideration of plane curves.

The Third Section is devoted to the general theory of curve surfaces, and of curves of double curvature; in the First Chapter of which are established the several forms of the equations of the tangent plane and normal line at any point of a curve surface, and of the linear tangent and normal plane at any point of a curve of double curvature.

In the Second Chapter the theory of conical and cylindrical surfaces is discussed, as also that of surfaces of revolution; and that remarkable case is examined, where the revolution of a straight line produces the same surface as the revolution of the hyperbola, to which this line is an asymptote. Throughout this chapter are interspersed many valuable and interesting applications of the calculus, chiefly from *Monge*. The Third Chapter embraces the theory of the curvature of surfaces in general, and will be found to form a collection of very beautiful theorems, the results, principally, of the researches of *Euler*, *Monge*, and *Dupin*. Most of these theorems have, however, usually been established by the aid of the *infinitesimal calculus*, or by the use of some other equally objectionable principle; they are here fairly deduced from the principles of the differential calculus, without, in any instance, departing from those principles, as laid down in the preliminary chapter. Those who are familiar with these inquiries will find that I have obtained some of these theorems in a manner much more simple and concise than has hitherto been done. I need only mention here, as instances of this simplicity, the theorems of *Euler* and of *Meusnier*, at pages 182 and 186.

The Fourth Chapter is on *twisted surfaces*, a class of surfaces which have never been treated of, to any extent, by any English author, although, as has been recently shown, the English were the first who noticed the peculiarities of certain individual surfaces belonging to this extensive class.* For what is here given, I am indebted to the French mathematicians, to *Monge* principally, and also to the *Chevalier Le Roy*, who has recently published a very neat and comprehensive little treatise on curves and surfaces.

The Fifth Chapter treats on the developable surfaces, or those which, like the cone and cylinder, may, if flexible, be unrolled upon a plane, without being twisted or torn. The Sixth Chapter is on curves of double curvature; and the Seventh, which concludes the volume, contains a few miscellaneous propositions intimately connected with the theory of surfaces. From the foregoing brief analysis, it will appear evident to those familiar with the present state of mathematical instruction in this country, that I have introduced, into a little duodecimo volume, a more comprehensive view of the theory and applications of the differential calculus than has yet appeared in the English language. But I have aimed at more than this; I have endeavoured to simplify and improve much that I have adopted from foreign sources; and, above all, to establish every thing here taught, upon principles free from inconsistency and logical objections; and if it be found, upon examination, that I have entirely failed in this endeavour, I shall certainly feel a proportionate disappointment.

I am not, however, so sanguine as to look for much public encouragement of my labours, however successfully they may have been devoted: it is not customary to place much value, in this country, upon any mathematical production, of whatever merit, that does not emanate from CAMBRIDGE. The hereditary reputation enjoyed by this University, and bequeathed to it by the genius of *Barrow*, of *Newton*, and of *Cotes*, seems to have endowed it with such strong claims on the public attention and respect, that every thing it puts forth is always received as the best of its kind. If this be the case with Cambridge books, of course it is also the case with Cambridge men, and accordingly we find almost all our public mathematical situations filled by members of this University. It is true that now

* See *Leybourn's Repository*, No. 22.

and then, in the course of half a century, we find an exception to this; one or two instances on record have undoubtedly occurred, where it has been, by some means or other, discovered that men who had never seen Cambridge knew a *little* of mathematics, as in the case of *Thomas Simpson*, and of *Dr. Hutton;* but such instances are rare. It is not for me to inquire into the justice of this exclusive system; but, while such a system prevails, there need be little wonder at the decline of science in England: while all inducement to cultivate science is thus confined to a particular set of men, no wonder that its votaries are few. It is to be hoped, however, that in the present "liberal and enlightened age," such a state of things will not long continue, and that even the poor and unfriended student may be cheered up, amidst all the obstacles that surround him, in the laborious and difficult, but sublime and elevating career on which he has entered, by a well-founded assurance that his exertions, if successful, will not be the less appreciated because they were solitary and unassisted.

May 12, 1831.

J. R. YOUNG.

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ERRATA.

PAGE 33, For article 20, *read* article 23.

183, art. 157, at bottom of page, *for* $\frac{1}{r} = r'$, *read* $\frac{1}{R} = t'$.

184, articles 158, 159, 160, should not be numbered.

THE

DIFFERENTIAL CALCULUS.

SECTION I.

ON THE

DIFFERENTIATION OF FUNCTIONS IN GENERAL.

CHAPTER I.

EXPLANATION OF FIRST PRINCIPLES.

Article (1.) ALL quantities which enter into calculation, may be divided into two principal classes, *constant* quantities and variable quantities; the former class comprehending those which undergo no change of value, but remain the same throughout the investigation into which they enter; while those quantities which have no fixed or determinate value, constitute the latter class.

In algebra we usually employ the first letters, a, b, c, &c. of the alphabet, to represent *known* quantities, and the latter letters, z, y, x, &c. as symbols of the unknown quantities; but, in the higher calculus, the early letters are adopted as the symbols of *constant* quantities, whether they be known or unknown, and the latter letters are used to represent *variables.*

Any analytical expression composed of constants and variables, is said to be a *function* of the variables. Thus, if $y + ax^2 + bx + c$, then is y a function of x, because x enters into the expression for y;

y is also a function of x in the expressions $y = a^x + b$, $y = \log. x + ax^2$, &c. and, as in each of these cases the form of the function is exhibited, y is said to be an *explicit* function of x; but, in such equations as

$$ax^2 + by^2 + cxy + x + y + e = 0, x^3 + x^2y - ay^2 = y^3 + bx + c, \&c.$$

where the form of the function that y is of x, can be ascertained only by solving the equation, y is an *implicit* function of x.

Similar remarks apply to the equations

$$z = ax^2 + by^2 + cx + e = 0, az^2 + by^2 + cxz + e = 0, \&c.$$

z being an explicit function of x and y in the first, and an implicit function of the same variables in the second equation.

If we wish to express that y is an explicit function of x, without writing the form of that function, we adopt the notation $y = \mathrm{F}x$, or $y = fx$, or $y = \varphi x$, &c. and, to denote an implicit function, we write $\mathrm{F}(x, y) = 0, f(x, y) = 0$, &c.*

(2.) Let us now examine the effect produced on the function y, by a change taking place in the variable x, and, for a first example, let us take the equation $y = mx^2$. Changing, then, x into $x + h$, and representing the corresponding value of y by y', we have

$$y' = m (x + h)^2$$

or, by developing the second number,

$$y' = mx^2 + 2mxh + mh^2.$$

As a second example, let us take the equation $y = x^3$, and putting as before y' for the value of the function, when x is changed into $x + h$, we have

$$y' = (x + h)^3 = x^3 + 3x^2h + 3xh^2 + h^3.$$

We thus see, in these two examples, the effect produced on the function by changing the value of the variable, and, on account of this dependence of the value of the function upon that of the variable, the former, that is y, is called the *dependent* variable, and the latter, x, the *independent* variable.

* In this general mode of expression, F, f, and ϕ, are mere symbols, representing the words *a function of*: thus, Fx, or fx, means a function of x, the form of the latter differing from that of the former. *Ed.*

Let us now ascertain the difference of the values of each of the above functions of x, in the two states y and y'. In the first example,

$$y' - y = 2mxh + mh^2.$$

In the second,

$$y' - y = 3x^2h + 3xh^2 + h^3,$$

so that, in the equation $y = mx^2$, if h be the increment of the variable x, we see that $2mxh + mh^2$ will be the corresponding increment of the function y; and, in the equation $y = x^3$, if x take the increment h, the corresponding increment of the function will be $3x^2h + 3xh^2 + h^3$.

We may, therefore, in each of these cases, readily find an expression for the ratio of the increment of the function to that of the variable, that is to say, the value of the fraction $\frac{y' - y}{h}$.

In the first case,

$$\frac{y' - y}{h} = 2mx + h.$$

In the second,

$$\frac{y' - y}{h} = 3x^2 + 3xh + h^2.$$

It is here worthy of remark, that in both these expressions for the ratio, the first term is independent of h; so that, however we alter the value of h, this first term will remain unchanged. If, therefore, h be supposed to diminish continually, and, at length, to become 0, the said first term will then express the value of the ratio. This first term, then, is the *limit* to which the ratio approaches as h diminishes, but which limit it cannot attain till h becomes absolutely 0.

In the first of the foregoing examples, $2mx$ is the limit of the ratio $\frac{y' - y}{h}$; or it is the value towards which this ratio continually approaches when h is continually diminished, and to which it ultimately arrives when these continual diminutions bring it at length to $h = 0$. In the second example the limit is $3x^2$.

(3.) We may now understand what is meant by the *limit of the ratio of the increment of the function to that of the variable.* It is the determination of this limit, in every possible form of the function, that is the principal object of the differential calculus. The limit itself is

called the *differential coefficient*, derived from the function; so that, if the function be mx^2, the differential coefficient, as we have seen above, is $2mx$, and the differential coefficient derived from the function, x^3, is $3x^2$.

In both these cases, as indeed in every other, the respective differential coefficients are only so many particular values of the general symbol $\frac{0}{0}$, to which $\dfrac{y'-y}{h}$ always reduces when $h=0$. In the first example above, $\frac{0}{0}=2mx$; in the second, $\frac{0}{0}=3x^2$.

Instead of the general symbol $\frac{0}{0}$, a particular notation is employed to represent the limiting ratio, or differential coefficient, in each particular case; thus, if y is the function, and x the independent variable, the differential coefficient is represented thus, $\dfrac{dy}{dx}$. If z were the function, and y the independent variable, the differential coefficient would be $\dfrac{dz}{dy}$; the expressions $\dfrac{dy}{dx}$ and $\dfrac{dz}{dy}$ have, we see, the advantage over the symbol $\frac{0}{0}$, of particularizing the function and the independent variable under consideration, and this, it must be remembered, is all that distinguishes $\dfrac{dy}{dx}$ or $\dfrac{dz}{dy}$ from $\frac{0}{0}$, for dy, dz, dx, are each absolutely 0.

This notation being agreed upon, we have, when $y=mx^2$,

$$\frac{dy}{dx}=2mx,$$

and, when $y=x^3$,

$$\frac{dy}{dx}=3x^2.$$

As a third example, let the function $y=a+3x^2$ be proposed, then, changing x into $x+h$ and y into y', we have

$$y'=a+3x^2+6xh+3h^3,$$

$$\therefore \frac{y'-y}{h}=6x+3h,$$

and, making $h=0$, we have, for the differential coefficient,

$$\frac{dy}{dx}=6x.$$

If in this example the function had been $3x^2$, instead of $a+3x$, the differential coefficient would obviously have been the same.

As a fourth example, let $y = ax^2 \pm b$,

$$\therefore y' = ax^2 \pm b + 2axh + ah^2,$$

$$\therefore \frac{y' - y}{h} = 2ax + ah \therefore \frac{dy}{dx} = 2ax,$$

which would have been the same if the constant b had not entered the function.

As a last example, take the function $y - (a + bx)^2$ or $y - a^2 + 2abx + b^2x^2$, which, when x is changed into $x + h$, becomes

$$y' = a^2 + 2ab\,(x + h) + b^2\,(x + h)^2$$
$$= a^2 + 2abx + b^2x^2 + 2\,(ab + b^2x)\,h + b^2h^2,$$

$$\therefore \frac{y' - y}{h} = 2b\,(a + bx) + b^2h,$$

$$\therefore \frac{dy}{dx} = 2b\,(a + bx).$$

It should be remarked, that of the two parts dy, dx, of which the symbol $\frac{dy}{dx}$ consists, the former is called the differential of y, and the latter the differential of x. These differentials, although each $= 0$, have, nevertheless, as we have already seen, a determinate relation to each other; thus, in the last example, this relation is such, that $dy = 2b\,(a + bx)\,dx$, and, although this is the same as saying that $0 = 2b\,(a + bx) \times 0$; yet, as we can always immediately obtain from this form the true value of $\frac{0}{0}$ or $\frac{dy}{dx}$, we do not hesitate occasionally to make use of it.

From the expression for the differential of a function, we readily see the propriety of calling $\frac{dy}{dx}$ a *coefficient*, being, indeed, the coefficient of dx.

CHAPTER II.

DIFFERENTIATION OF FUNCTIONS OF ONE VARIABLE.

(4.) Let fx represent any function of x whatever, then, if x be chan-

ged into $x + h$, the general form of the development of $f(x + h)$, arranged according to the powers of h, will be

$$f(x + h) = fx + Ah + Bh^2 + Ch^3 + Dh^4 + \&c.$$

as may be proved as follows:

1. *The first term of the development must be fx.* This is obvious, for this first term is what the whole development reduces to when $h = 0$, but we must in this case have the identity $fx = fx$; hence, fx is the first term.

2. *None of the exponents of h can be fractional.* For if any exponent were fractional, the term into which it enters would be irrational, so that the development of $f(x + h)$, containing an irrational term, $f(x + h)$ itself would be irrational, since it is impossible that there should be an equality between an irrational expression and one that is rational. But, if $f(x + h)$ were irrational, fx would likewise be irrational, for the former function differs from the latter only in this, that $x + h$ occupies in it the place of x in the latter; and, therefore, as many values as fx has, in consequence of the radicals that may enter into it, so many values, and no more, must $f(x + h)$ have. As, therefore, the first term of the development of $f(x + h)$ has the same number of values as $f(x + h)$ itself, none of the succeeding terms can contain a fractional power of h; for, otherwise, the development would have more values than its equivalent function, which is absurd.

3. *None of the exponents of h can be negative* in the general development. For, on the supposition that a negative power of h enters any term, that term would become infinite when $h = 0$; but, when $h = 0$, the function is simply fx, which is not necessarily infinite; so that, that development into which a negative power of h enters, cannot be the *general development* of $f(x + h)$, understanding, by the general development, that which does not restrict x to any particular value or values. By supposing a negative power of h to appear in the development, we have just seen that such development would restrict x to the values determined, by the equation $fx = \infty$, or by the equation $\frac{1}{fx} = 0$, to the exclusion of all other values, and is, therefore, not general.*

* It has been shown, that the general development of fx, when x becomes $x + h$, is $f(x + h) = fx + Ah^a + Bh^b + Ch^c + \&c.$, in which the exponents a, b, c, &c. of the increment h of the variable, are whole and positive numbers, it

As to the method of determining the coefficients A, B, C, &c. that will be investigated hereafter (*in Chapter* iv.)

4. It must be here remarked, that, although the general development of every functiou of x is of the above form, yet we are not to suppose that this form will remain unchanged whatever *particular* value we give to x, for particular values may be so chosen as to render this form of development impossible, and such impossibility will be intimated by the assumed value of x rendering some of the coefficients A, B, C, &c. infinite. The well known binomial theorem, which we already know to be of the above form, will afford an illus-

yet remains to prove, that they will be represented by the series 1, 2, 3, &c. as assumed in the commencement of this chapter.

It is already known, that the first term of the development of $f(x+h)$, is the primitive function, fx, and that the remainder must disappear when $h=0$, which remainder must then contain h as a factor, so that we shall have

$$f(x+h)=fx+Ph \;.\;.\;.\;.\; (1),$$

and thence

$$P=\frac{f(x+h)-fx}{h}.$$

P being a new function of x and h, can likewise be developed, of which the first term will be the value P will assume when $h=0$, which we will represent by p, and as the remaining terms must vanish when $h=0$, they must contain h as a factor; we thus have

$$P=p+Qh,$$

which substituted in equation (1), gives

$$f(x+h)=fx+ph+Qh^2, \&c. \;.\;.\;.\;.\; (2)$$

again we have

$$Q=\frac{P-p}{h};$$

in which Q is another function of x and h, and may be developed the same as P; the first term of this development will be the value of Q when $h=0$, which we represent by q, and the remainder becoming zero when $h=0$, must contain h as a factor; we shall then have

$$Q=q+Rh,$$

in which R is another function of x and h. Expression (2), will thus become

$$f(x+h)=fx+ph+qh^2+Rh^3+\&c.$$

being the general form of the development of $f(x+h)$, in which p, q, r, &c. are functions of x alone, and correspond to the coefficients A, B, C, &c. in the article under consideration. *Ed.*

tration of this. Thus the general development of $fx = \sqrt{x + a}$, when we replace x by $x + h$, is, by the above mentioned theorem,

$$f\{(x + h) + a\} = \sqrt{(x + a) + h} =$$
$$(x + a)^{\frac{1}{2}} + \tfrac{1}{2}(x + a)^{-\frac{1}{2}} h - \tfrac{1}{8}(x + a)^{-\frac{3}{2}} h^2 + \&c.$$

where, in the case $x = - a$, all the coefficients become infinite, and the development, according to the positive integral powers of h, becomes in this case impossible; for the function then becomes merely $\sqrt{h}$ or $h^{\frac{1}{2}}$, in which the exponent of h is fractional. The impossibility of the proposed form of development in such particular case is always intimated, as in the example just adduced, by the circumstance of infinite coefficients entering it, for imaginary coefficients would imply merely that the function $f(x + h)$ for the assumed value of x becomes imaginary, and not that the development failed. A particu-ar examination of the cases in which the general form of the deve-.opment fails to have place, will form the subject of a future chapter; at present it is sufficient to apprise the student that such failing cases may exist.

(5.) By transposing the first term in the general development of $f(x + h)$, we have

$$f(x + h) - fx = \mathrm{A}h + \mathrm{B}h^2 + \mathrm{C}h^3 + \&c.$$
$$\therefore \frac{f(x + h) - fx}{h} = \mathrm{A} + \mathrm{B}h + \mathrm{C}h^2 + \&c.$$

hence, when $h = 0$,

$$\frac{dfx}{dx} = \mathrm{A},$$

from which result we learn, that *the coefficient of the second term, in the development of the function $f(x + h)$, is the differential coefficient derived from the function fx*; so that the finding the differential coefficient from any proposed function, fx, reduces itself to the finding the coefficient of the second term in the general development of $f(x + h)$, or of the first term in the developed difference $f(x + h) - fx$.

Having obtained this general result, we may now proceed to apply it to functions of different terms; but it will be proper previously to observe, that those constants which are connected with the variable in the function fx, only by way of addition or subtraction, cannot appear in the coefficient A; because A, being multiplied by h, can contain

no quantities which are not among those multiplied by $x + h$ in $f(x + h)$, or by x in fx.

(6.) *To differentiate the product of two or more functions of the same variable.*

Let y, z, be functions of x, in the expression

$$u = ayz.$$

By changing x into $x + h$, the function y becomes

$$y' = y + Ah + Bh^2 + Ch^3 + \&c. \ . \ . \ . \ (1),$$

and the function z becomes

$$z' = z + A'h + B'h^2 + C'h^3 + \&c. \ . \ . \ . \ (2).$$

Hence, when $h = 0$, we have from (1)

$$\frac{y' - y}{h} = \frac{dy}{dx} = A,$$

and from (2)

$$\frac{z' - z}{h} = \frac{dz}{dx} = A'.$$

Multiplying the product of (1) and (2) by a, we have

$$u' = ayz + a\,(Az + A'y)\,h + \&c.*$$

$$= ayz + a\,(\frac{dy}{dx}z + \frac{dz}{dx}y)\,h + \&c.$$

therefore, $a\left(\frac{dy}{dx}z + \frac{dz}{dx}y\right)$ being the coefficient of the second term of the development of u', we have

$$\frac{du}{dx} = az\frac{dy}{dx} + ay\frac{dz}{dx}$$

$$\therefore du = azdy + aydz \ . \ . \ . \ (3).$$

Hence, *to differentiate the product of two functions of the same variable, we must multiply each by the differential of the other, and add the results.*

It will be easy now to express the differential of a product of three functions of the same variable. Let.

$$u = wyz$$

be the product of three functions of x; then, putting v for wy, the expression is

$$u = vz;$$

hence, by (3),

* u' is that value which u attains when the functions y and z have varied by virtue of the variation h of the variable x on which they depend. *Ed.*

$$du = zdv + vdz,$$

but $v = wy$; therefore, by (3), $dv = ydw + wyd$; consequently, by substitution,

$$du = zydw + zwdy + wydz \;.\;.\;.\; (4),$$

and it is plain that in this way the differential may be found, be the factors ever so many; so that, generally, *to differentiate a product of several functions of the same variable, we must multiply the differential of each factor by the product of all the other factors, and add the results.*

If we suppose the factors to be all equal to each other, we shall obtain a rule to differentiate a positive integral power. Thus the differential of the function

$$u = x^m = x \cdot x \cdot x \cdot x \ldots .$$

is

$$du = x^{m-1}\,dx + x^{m-1}\,dx + x^{m-1}\,dx + \&\text{c. to } m \text{ terms},$$

that is

$$du = mx^{m-1}\,dx \;\therefore\; \frac{du}{dx} = mx^{m-1}.$$

This form of the differential is preserved whether m be integral or fractional, positive or negative; but, to prove this, we must first differentiate a fraction.

(7.) *To differentiate a fraction.* Let $u = \dfrac{y}{z}$, y and z being functions of x; therefore $uz = y$, and $duz = dy$, that is, by the last article,

$$zdu + udz = dy \;\therefore\; du = \frac{dy - udz}{z},$$

or, substituting $\dfrac{y}{z}$ for u,

$$du = \frac{zdy - ydz}{z^2}.$$

Hence, to differentiate a fraction, the rule is this: *From the product of the denominator, and differential of the numerator, subtract the product of the numerator, and differential of the denominator, and divide the remainder by the square of the denominator.*

(8.) *To differentiate any power of a function.*

The form of the differential when the power is whole and positive has been already established. Let then

$$u = y^{\frac{m}{n}}$$

be proposed, y being a function of x, and $\frac{m}{n}$ being a positive fraction.

Since $u^n = y^m$,

$$\therefore nu^{n-1}\,du = my^{m-1}\,dy,$$

$$\therefore du = \frac{m}{n} \cdot \frac{y^{m-1}}{u^{n-1}}\,dy = \frac{m}{n} \cdot \frac{y^{m-1}}{y^{\frac{mn-m}{n}}}\,dy.$$

Now

$$(m-1) - \frac{mn-m}{n} = \frac{m}{n} - 1,$$

consequently,

$$du = \frac{m}{n}\,y^{\frac{m}{n}-1}\,dy.$$

Let now the exponent be negative, or

$$u = y^{-\frac{m}{n}}$$

$$\therefore u^n = y^{-m} = \frac{1}{y^m}$$

$$\therefore du^n = d\frac{1}{y^m}$$

but

$$du^n = nu^{n-1}du, \text{ and } d\frac{1}{y^m} = -m\frac{y^{m-1}}{y^{2m}}\,dy = -my^{-m-1}\,dy,$$

$$\therefore nu^{n-1}\,du = -my^{-m-1}\,dy,$$

$$\text{and } du = -\frac{my^{-m-1}}{nu^{n-1}}\,dy,$$

or, substituting for u its equal $y^{-\frac{m}{n}}$, we have

$$du = -\frac{m}{n}\,y^{-\frac{m}{n}-1}\,dy.$$

Hence, generally, *to differentiate a power, we must multiply together these three factors, viz. the index of the power, the power itself diminished by unity, and the differential of the root.*

This rule might have been deduced with less trouble, by availing ourselves of the binomial theorem, for, supposing in $u = y^p$ that the increment of the function y becomes k when the increment of x becomes h, we have $u' = (y + k)^p$ and, by the binomial theorem, the

coefficient of the second term of the expansion of $(y + k)^p$ is py^{p-1}, whether p be positive or negative, whole or fractional. As, however, we propose to demonstrate the binomial theorem by means of the differential calculus, we have thought it necessary to establish the fundamental principles of differentiation, independently of this theorem.

(9.) If it be required to differentiate an expression consisting of several functions of the same variable, combined by addition or subtraction, it will be necessary merely to differentiate each separately, and to connect together the results by their respective signs. For let the expression be

$$u = aw + by + cz + \&c.$$

in which w, y, z, are functions of x. Then, changing x into $x + h$ and developing,

$$\begin{array}{lll} w & \text{becomes} & w + Ah + Bh^2 + \&c. \\ y & & y + A'h + B'h^2 + \&c. \\ z & & z + A''h + B''h^2 + \&c. \\ \therefore u & & u + (aA + bA' + cA'' + \&c.)\,h + \&c. \end{array}$$

$$\therefore du = aAdx + bA'dx + cA''dx + \&c.$$

But

$$Adx = dw,\ A'dx = dy,\ A''dx = dz,\ \&c.$$

therefore

$$du = adw + bdy + cdz + \&c.$$

that is, *the differential of the sum of any number of functions is equal to the sum of their respective differentials.*

(10.) We shall now apply the foregoing general rules to some examples.

EXAMPLES.

1. Let it be required to differentiate the function

$$y = 8x^4 - 3x^3 - 5x.$$

By the rule for powers (8) the differential of $8x^4$ is $8 \times 4x^3dx$, and the differential of $-3x^3$ is $-3 \times 3x^2dx$; also the differential of $-5x$ is $-5dx$; hence (9),

$$dy = 32x^3dx - 9x^2dx - 5dx,$$

$$\therefore \frac{dy}{dx} = 32x^3 - 9x^2 - 5.$$

2. Let $y = (x^3 + a)(3x^2 + b)$.

By the rule for differentiating a product (6), we have

$$dy = (x^3 + a)\, d\,(3x^2 + b) + (3x^2 + b)\, d\,(x^3 + a),$$

and (8),

$$d\,(3x^2 + b) = 6xdx,\ d\,(x^3 + a) = 3x^2dx,$$

$$\therefore\ dy = (x^3 + a)\, 6xdx + (3x^2 + b)\, 3x^2dx,$$

$$\therefore\ \frac{dy}{dx} = 15x^4 + 3x^2b + 6ax.$$

3. Let $y = (ax + x^2)^2$.

The differential of the root $ax + x^2$ of this power, is $adx + 2xdx$, therefore,

$$dy = 2\,(ax + x^2)\,(a + 2x)\,dx,$$

$$\therefore\ \frac{dy}{dx} = 2\,(ax + x^2)\,(a + 2x).$$

4. Let $y = \sqrt{a + bx^2}$.

The differential of the root or function under the radical, is $2bxdx$; hence

$$dy = \tfrac{1}{2}\,(a + bx^2)^{-\frac{1}{2}}\, 2bxdx = \frac{bx}{\sqrt{a + bx^2}}\,dx,$$

$$\therefore\ \frac{dy}{dx} = \frac{bx}{\sqrt{a + bx^2}}.$$

5. Let $y = (a + bx^m)^n$.

The differential of the root or function within the parenthesis, is $mbx^{m-1}\,dx$; hence

$$dy = n\,(a + bx^m)^{n-1}\, mbx^{m-1}\,dx,$$

$$\therefore\ \frac{dy}{dx} = bmn\,(a + bx^m)^{n-1}\, x^{m-1}.$$

6. Let $y = \dfrac{x^2}{(a + x^3)^2}$.

The differential of the numerator of this fraction is $2xdx$, and the differential of $a + x^3$ is $3x^2dx$, therefore the differential of the denominator is $2(a + x^3)\,3x^2dx$; hence (7),

$$dy = \frac{(a + x^3)^2\, 2xdx - 6x^4\,(a + x^3)dx}{(a + x^3)^4} = \frac{2ax - 4x^4}{(a + x^3)^3}\,dx,$$

$$\therefore\ \frac{dy}{dx} = \frac{2x(a - 2x^3)}{(a + x^3)^3}.$$

7. Let $y = \{a + \sqrt{(b + \frac{c}{x^2}}\}^4$.

The differential of the root $a + \sqrt{(b + \frac{c}{x^2})}$ is $\frac{1}{2}(b + \frac{c}{x^2})^{-\frac{1}{2}}$

$d\frac{c}{x^2}$, and $d\frac{c}{x^2} = -\frac{2c}{x^3}dx$; hence

$$\frac{dy}{dx} = -4\{a + \sqrt{(b' + \frac{c}{x^2})}\}^3(b + \frac{c}{x^2})^{-\frac{1}{2}}\frac{c}{x^3}$$

$$= -\frac{4c\{a + \sqrt{(b + \frac{c}{x^2})}\}^3}{x^3\sqrt{b + \frac{c}{x^2}}}.$$

8. Let $y = \sqrt{x^2 + \sqrt{a + x^2}}$.

The differential of $x^2 + \sqrt{a + x^2}$ is $2xdx + (a + x^2)^{-\frac{1}{2}}xdx$,

$$\therefore \frac{dy}{dx} = \frac{x}{\sqrt{x^2 + \sqrt{a + x^2}}} + \frac{x}{2\sqrt{(a+x^2)(x^2+\sqrt{a+x^2})}}.$$

9. Let $y = \dfrac{x}{\sqrt{a^2 + x^2} - x}$.

Multiplying numerator and denominator by $\sqrt{a^2 + x^2} + x$, the expression becomes

$$y = \frac{x^2}{a^2} + \frac{x}{a^2}\sqrt{a^2 + x^2},$$

$$\therefore dy = d\frac{x^2}{a^2} + \frac{\sqrt{a^2 + x^2}}{a^2}dx + \frac{x}{a^2}d\sqrt{a^2 + x^2},$$

$$\therefore \frac{dy}{dx} = \frac{2x}{a^2} + \frac{\sqrt{a^2 + x^2}}{a^2} + \frac{x^2}{a^2\sqrt{a^2 + x^2}}$$

$$= \frac{2x}{a^2} + \frac{a^2 + 2x^2}{a^2\sqrt{a^2 + x^2}}.$$

10. $y = a^2 - x^2 \therefore \dfrac{dy}{dx} = -2x.$

11. $y = 4x^3 - 2x^2 + 7x + 3 \therefore \dfrac{dy}{dx} = 12x^2 - 4x + 7.$

12. $y = (a + bx)\,x^3 \therefore \frac{dy}{dx} = 4bx^3 + 3ax^2.$

13. $y = (a + bx + cx^2 + \&c.)^m \therefore \frac{dy}{dx} = m\,(a + bx + cx^2$ $+ \&c.)^{m-1}\,(b + cx + \&c.)$

14. $y = (a + bx^2)^{\frac{5}{4}} \therefore \frac{dy}{dx} = \frac{5bx}{2}\sqrt[4]{a + bx^2}.$

15. $y = a + \frac{4\sqrt{x}}{3 + x^2} \therefore \frac{dy}{dx} = \frac{6\,(1 - x^2)}{(3 + x^2)^2\sqrt{x}}.$

16. $y = a + b\sqrt{x} - \frac{c}{x} \therefore \frac{dy}{dx} = \frac{b}{2\sqrt{x}} + \frac{c}{x^2}.$

17. $y = (ax^3 + b)^2 + 2\sqrt{a^2 - x^2}\,(x - b) \therefore \frac{dy}{dx} = 6ax^2$ $(ax^3 + b) + \frac{2(a^2 - 2x^2 + bx)}{\sqrt{a^2 - x^2}}.$

18. $y = \frac{x}{x + \sqrt{1 - x^2}} \therefore \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}(1 + 2x\sqrt{1 - x^2})}.$

The functions in these examples are all *algebraic*, we shall now consider

Transcendental Functions.

(11.) Transcendental functions are those in which the variable enters in the form of an exponent, a logarithm, a sine, &c. Thus, a^x, a log. x, sin. x, &c. are transcendental functions: the first is an *exponential* function, the second a *logarithmic* function, and the third a *circular* function.

*To find the Differential of a Logarithm.**

(12.) Let it be required to differentiate log. x.

Put a for the base of the system of logarithms used, and let

$$M = \frac{1}{a - 1 - \frac{1}{2}(a - 1)^2 + \frac{1}{3}(a - 1)^3 - \&c.}\dagger$$

then

$$\log.\,(1 + n) = M\,(n - \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3 - \&c.)$$

or, putting $\frac{h}{x}$ for n,

* Note (A′).

† Algebra, Chap. vii. p. 219., or vol. i. p. 155, Lacroix's large work on the Differential Calculus. *Ed.*

$$\log.\ \frac{x+h}{x} = \log.\ (x+h) - \log. x = \mathrm{M}\left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \&c.\right)$$

$$\therefore\ \frac{\log.\ (x+h) - \log.\ x}{h} = \mathrm{M}\left(\frac{1}{x} - \frac{h}{2x^2} + \frac{h^2}{3x^3} - \&c.\right)$$

This is the general expression for the ratio of the increment of the function to that of the variable. Hence, taking the limit of this ratio, we have

$$\frac{d \log.\ x}{dx} = \frac{\mathrm{M}}{x} \ldots (1).$$

If the logarithms employed be hyperbolic $\mathrm{M} = 1$, and then

$$\frac{d \log.\ x}{dx} = \frac{1}{x} \ldots (2).$$

If they are not hyperbolic, write Log. instead of log. for distinction sake, then, since by putting a for $1 + n$ in the series for log. $(1 + n)$, we have

$$\log.\ a = a - 1 - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c. = \frac{1}{\mathrm{M}}$$

it follows, from the expression (1), that

$$\frac{d\,\mathrm{Log}.\ x}{dx} = \frac{1}{\log.\ a\ .\ x}.$$

Unless the contrary is expressed, the differential is always taken according to the hyperbolic system, because the expression is then simpler, log. a being $= 1$.

From the preceding investigation we learn, that *the differential of a logarithmic function is equal to the differential of the function divided by the function itself.*

(13.) *To differentiate an exponential function.*

1. Let $y = a^x$ then log. $y = x$ log. a $\therefore$ d log. $y = dx$ log. a, that is, $\frac{dy}{y} = dx \log a \therefore dy = y \log.\ a.dx = \log.\ a\ .\ a^x\, dx$.

Hence, *to differentiate an exponential, we must multiply together the hyp. log. of the base, the exponential itself, and the differential of the variable exponent.*

EXAMPLES.

1. Let $y = x(a^2 + x^2)\sqrt{a^2 - x^2}$ $\therefore$ log. $y =$ log. $x +$ log. $(a^2 + x^2) + \frac{1}{2}$ log. $(a^2 - x^2)$,

$$\therefore \frac{dy}{y} = \frac{dx}{x} + \frac{2xdx}{a^2 + x^2} - \frac{xdx}{a^2 - x^2} = \frac{a^2 + a^2x^2 - 4x^4}{x\,(a^2 + x^2)\,(a^2 - x^2)}\,dx,$$

therefore, substituting for y its value, we have,

$$\frac{dy}{dx} = \frac{a^4 + a^2x^2 - 4x^4}{\sqrt{a^2 - x^2}}.$$

2. $y = \log. \dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}.$

Multiplying numerator and denominator by the denominator, the expression becomes

$$y = \log. \frac{2x}{2a - 2\sqrt{a^2 - x^2}} = \log. x - \log. (a - \sqrt{a^2 - x^2})$$

$$\therefore \frac{dy}{dx} = \left\{\frac{1}{x} - \frac{x}{a\sqrt{a^2 - x^2} - a^2 + x^2}\right\} = \frac{a\sqrt{a^2 - x^2} - a^2}{x\sqrt{a^2 - x^2}\{a - \sqrt{a^2 - x^2}\}}$$

$$= \frac{-a}{x\sqrt{a^2 - x^2}}.$$

3. $y = \dfrac{\sqrt{x^2 + 2ax}}{\sqrt[3]{x^3 + x^2 - x}} \therefore \log. y = \frac{1}{2} \log. (x^2 + 2ax) - \frac{1}{3}$ $\log. (x^3 + x^2 - x),$

$$\therefore \frac{dy}{y} = \frac{x + a}{x^2 + 2ax}\,dx - \frac{3x^2 + 2x - 1}{3\,(x^3 + x^2 - x)}\,dx =$$

$$\frac{(1 - 3a)\,x^2 - (a + 2)\,x - a}{3x\,(x^2 + x - 1)\,(x + 2a)}\,dx,$$

$$\therefore \frac{dy}{dx} = \frac{\{(1 - 3a)\,x^2 - (a + 2)\,x - a\}\sqrt{x}}{3x^{\frac{4}{3}}\,(x^2 + x - 1)^{\frac{4}{3}}\,(x + 2a)^{\frac{1}{2}}}.$$

4. $y = x^{m\sqrt{-1}} \therefore \log. y = m\sqrt{-1} \log. x \therefore \dfrac{dy}{y} = m\sqrt{-1}\,\dfrac{dx}{x},$

$$\therefore \frac{dy}{dx} = m\frac{y}{x}\sqrt{-1} = m\sqrt{-1} \,.\, x^{m\sqrt{-1}-1}.$$

From this example it appears, that the rule at (8) *applies when the exponent is imaginary.*

5. $y = a^{x^x}.$

In this example the variable exponent is x^x; hence, calling it z and taking the logarithms, we have

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$\log. z = x \log. x \therefore \frac{dz}{z} = \log. x dx + \frac{x dx}{x} \therefore dz = x^x(1+\log. x)dx;$

hence, by the rule,

$$\frac{dy}{dx} = \log. a \,.\, a^{x^x} \,.\, x^x\,(1 + \log. x).$$

6. $y = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$, where e is the base of the hyperbolic system,

$$\therefore dy = e^{x\sqrt{-1}}\sqrt{-1}\,dx - e^{-x\sqrt{-1}}\sqrt{-1}\,dx,$$

$$\therefore \frac{dy}{dx} = (e^{x\sqrt{-1}} - e^{-x\sqrt{-1}})\sqrt{-1}.$$

7. $y = \log. (\log. x)$.* Put z for $\log. x \therefore y = \log. z \therefore dy = \frac{dz}{z}$

but $dz = d \log. x = \frac{dx}{x} \therefore dy = \frac{dx}{x \log. x} \therefore \frac{dy}{dx} = \frac{1}{x \log. x}.$

8. $y = (\log. x^n)^m \therefore \frac{dy}{dx} = \frac{mn\,(\log. x^n)^{m-1}}{x}.$

9. $y = \log. \{(a+x)^m\,(a'+x)^{m'}\,(a''+x)^{m''}\} \therefore \frac{dy}{dx} = \frac{m}{a+x}$
$+ \frac{m'}{a'+x} + \frac{m''}{a''+x}.$

10. $y = \log. \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} - \sqrt{x}} \therefore \frac{dy}{dx} = \frac{\sqrt{a}}{(a-x)\sqrt{x}}.$

11. $y = e^{x^x} \therefore \frac{dy}{dx} = e^{x^x} \,.\, x^x\,(1 + \log. x).$

12. $y = (\log.)^n x \therefore \frac{dy}{dx} = \frac{1}{x \log. x\,(\log.)^2 x \ldots (\log.)^{n-1} x}.$

13. $\log. y = \frac{\sqrt{1+x^2}}{x} \therefore \frac{dy}{dx} = -\frac{1}{x^2}.$

14. $y = a^{b^z}$, z being a function of x, $\therefore \frac{dy}{dx} = \log. a \log. b \,.\, a^{b^z} b^z . \frac{dz}{dx}$

* This means the logarithm of the logarithm of x, but the notation we shall hereafter adopt will be $(\log.)^2 x$, and which we shall extend to circular functions; thus, instead of sin. (sin. x), we shall write $(\sin.)^2 x$, the *square* of the sine being written without the parenthesis, thus, $\sin.\,^2 x$. We may call such expressions as $(\log.)^n\,x$, $(\sin.)^n\,x$, &c. the nth log. of x, the nth sine of x, &c.

15. $y = a^{b^{x^2+x}} \therefore \dfrac{dy}{dx} = \log. a \,.\, \log. b \,.\, a^{b^{x^2+x}} \,.\, b^{x^2+x}(2x+1)$

16. $y = a^{\log. x} \therefore \dfrac{dy}{dx} = \dfrac{\log. a \,.\, a^{\log. x}}{x}.$

17. $y = e^{(\log.)^n x} \therefore \dfrac{dy}{dx} = \dfrac{e^{(\log.)^n x}}{\log. x\, (\log.)^2 x \,.\,.\,.\,.\, (\log.)^{n-1} x}.$

18. $y = x^{x^x} \therefore \dfrac{dy}{dx} = x^{x^x} \,.\, x^x \left\{\dfrac{1}{x} + \log. x\, (1 + \log. x)\right\}.$

(14.) *To differentiate circular functions.*

Let x represent the versed sine of an arc of a circle whose radius is r, then $r - x$ will represent the cosine of the same arc, and, by trigonometry,

$$\frac{\text{tan.}}{\text{sin}} = \frac{r}{r - x}.$$

In this expression, x is the independent variable, and as this diminishes, the arc itself diminishes, both vanishing simultaneously, and the ultimate ratio of $\dfrac{\text{tan.}}{\text{sin.}}$ is $\dfrac{r}{r} = 1$; that is, the sine and tangent of an arc approximate to each other as the arc diminishes, and at length become equal. As the arc is between the sine and tangent when these become equal, the arc, also, must become equal to each; therefore, we may conclude, that the ultimate ratios are as follows:

$$\frac{\text{tan.}}{\text{sin.}} = 1, \frac{\text{arc}}{\text{sin.}} = 1, \frac{\text{arc}}{\text{tan.}} = 1, \frac{\text{arc}}{\text{chord}} = 1, \frac{\text{sin.}}{\text{chord}} = 1; \frac{\text{tan.}}{\text{chord}} = 1.*$$

1. Let it now be required to find the differential of sin. x. Changing x into $x + h$, we have (*Gregory's Trig.* p. 48)

$$\sin. (x + h) = \sin. x + 2 \sin. \tfrac{1}{2} h \cos. (x + \tfrac{1}{2} h),$$

$$\therefore \frac{\sin. (x + h) - \sin. x}{h} = \frac{\sin. \frac{1}{2} h}{\frac{1}{2} h} \cos. (x + \tfrac{1}{2} h);$$

when $x = 0, \dfrac{\sin. \frac{1}{2} h}{\frac{1}{2} h} = 1, \therefore \dfrac{d \sin. x}{dx} = \cos. x \therefore d \sin. x = \cos. x dx.$

2. To differentiate cos. x.

$d \cos. x = d \sin. (\tfrac{1}{2} \pi - x) = - \cos. (\tfrac{1}{2} \pi - x)\, dx = - \sin. x\, dx.$

* The differentiation of circular functions may be obtained independently of these results. See the note (A) at the end of the volume.

Cor. As d cos. $= - d$ ver. sin. $\therefore d$ ver. sin. $x =$ sin. $x\,dx$.

3. To differentiate tan. x.

$$d \tan. x = d\frac{\sin. x}{\cos. x} = \frac{\cos. x\, d \sin. x - \sin. x\, d \cos. x}{\cos.^2 x},$$

that is

$$d \tan. x = \frac{\cos.^2 x + \sin.^2 x}{\cos.^2 x}\,dx = \frac{1}{\cos.^2 x}\,dx = \sec.^2 x\,dx.$$

4. To differentiate cot. x.

$$d \cot. x = d \tan. (\tfrac{1}{2}\pi - x) = - \sec.^2 (\tfrac{1}{2}\pi - x)\,dx = - \operatorname{cosec.}^2 x\,dx.$$

5. To differentiate sec. x.

$$d \sec. x = d\frac{1}{\cos.^2 x} = \frac{\sin. x}{\cos.^2 x}\,dx = \tan. x \sec. x\,dx.$$

6. To differentiate cosec. x.

$$d \operatorname{cosec.} x = d\frac{1}{\sin. x} = -\frac{\cos. x}{\sin.^2 x}\,dx = -\cot. x \operatorname{cosec.} x\,dx.$$

These six forms the student should endeavour to preserve in his memory.

EXAMPLES.

1. $y = \sin.^2 x \therefore dy = 2 \sin. x\, d \sin. x = 2 \sin. x \cos. x\,dx$ $= \sin. 2x\,dx$,

$$\therefore \frac{dy}{dx} = \sin. 2x.$$

2. $y = \sin.^n x \therefore dy = n \sin.^{n-1} x\, d \sin. x = n \sin.^{n-1} x \cos. x dx$,

$$\therefore \frac{dy}{dx} = n \sin.^{n-1} x \cos. x.$$

3. $y = \cos. mx \therefore dy = - \sin. mx dmx = - m \sin. mx dx$,

$$\therefore \frac{dy}{dx} = - m \sin. mx.$$

4. $u = y \tan. x^n$, y being a function of x, $\therefore du = \tan. x^n\,dy +$ $y\,d \tan. x^n$, now $d \tan. x^n = \sec.^2 x^n\,dx^n = nx^{n-1} \sec.^2 x^n\,dx$,

$$\therefore \frac{du}{dx} = \tan. x^n \frac{dy}{dx} + ynx^{n-1} \sec.^2 x^n.$$

5. $u = \cot. x^y \therefore du = - \operatorname{cosec.}^2 x^y\,dx^y$. Put $z = x^y \therefore \log.$ $z = y \log. x$,

$$\therefore \frac{dz}{z} = y\frac{dx}{x} + \log. xdy \therefore dz = dx^y = (y\frac{dx}{x} + \log. x\, dy)x^y,$$

$$\text{and } \frac{du}{dx} = -\text{cosec.}^2 x^y \left(\frac{y}{x} + \log. x\,\frac{dy}{dx}\right) x^y.$$

6. $y = xe^{\cos. x} \therefore dy = e^{\cos. x}dx + xe^{\cos. x} d \cos. x = e^{\cos. x}$ $(1 - x \sin. x)\, dx,$

$$\therefore \frac{dy}{dx} = e^{\cos. x}(1 - \sin. x).$$

7. $y = \log. (x\,e^{\cos. x}) \therefore dy = \dfrac{d\,(x\,e^{\cos. x})}{x\,e^{\cos. x}}$, and $d\,(x\,e^{\cos. x}) =$ $e^{\cos. x}(1 - x \sin. x)dx,$

$$\therefore \frac{dy}{dx} = \frac{1 - x \sin. x}{x}.$$

8. $y = \cos. x + \sin. x\,\sqrt{-1},\ \therefore \dfrac{dy}{dx} = -\sin. x + \cos. x\sqrt{-1}$

9. $y = \cos. x + \cos. 2x + \cos. 3x + \&c. \therefore \dfrac{dy}{dx} = -\ (\sin. x$ $+ 2 \sin. 2x + 3 \sin. 3x + \&c.)$

10. $y = x\,e^{\tan. x} \therefore \dfrac{dy}{dx} = \{1 + x \sec.^2 x\}\, e^{\tan. x}.$

11. $y = \dfrac{\sin.^m x}{\cos.^n x} \therefore \dfrac{dy}{dx} = m\left\{\dfrac{\sin.^{m-1} x}{\cos.^{n-1} x}\right\} + n\left\{\dfrac{\sin.^{m+1} x}{\cos.^{n+1} x}\right\}.$

12. $u = \sec. x^y \therefore \dfrac{du}{dx} = \tan. x^y \sec. x^y\, x^y\left\{\dfrac{y}{x} + \log. x\,\dfrac{dy}{dx}\right\}.$

(15.) In the preceding trigonometrical expressions, the arc is considered as the independent variable, and the lines sine, cosine, &c. as functions of it; we shall now consider the *inverse functions* as they are called, that is, those in which the arc is considered as a function of the sine, the cosine, &c. A particular notation has been proposed for inverse functions: thus, if $y = \mathrm{F}x$ be the direct function, then $x = \mathrm{F}^{-1}\, y$ is the inverse function, that is, if we represent the function that y is of x by $y = \mathrm{F}x$, the function that x is of y will be denoted by $x = \mathrm{F}^{-1}\, y$. By thus representing these inverse functions, we may return immediately to the direct functions, considering, for the moment, F^{-1} in the light of a negative power of F, or an equivalent to $\dfrac{1}{\mathrm{F}}$; for then $x = \mathrm{F}^{-1}\, y$ immediately leads to $y = \mathrm{F}x$.* Thus, if

* See note (B′).

$x = \log.^{-1} y$ $\therefore$ $y = \log. x$, the inverse function $\log.^{-1} y$ meaning *the number whose log. is y*. In like manner, $y = \sin.^{-1} x$ means that y is the arc whose sine is x; that is, returning to the direct function, $\sin. y = x$.

1. To differentiate $y = \sin.^{-1} x$.

Here the direct function is $\sin. y = x$ $\therefore$ $d \sin. y = dx$, that is,

$$\cos. y dy = dx \therefore \frac{dy}{dx} = \frac{1}{\cos. y} = \frac{1}{\sqrt{1 - \sin.^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

2. To differentiate $y = \cos.^{-1} x$.

$$\cos. y = x \therefore \sin. y dy = dx \therefore \frac{dy}{dx} = -\frac{1}{\sin. y} = -\frac{1}{\sqrt{1 - \cos.^2 y}}.$$

3. To differentiate $y = \operatorname{versin}.^{-1} x$.

$$\operatorname{versin}. y = x \therefore \sin. y dy = dx,$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sin. y} = \frac{1}{\sqrt{2x - x^2}}.$$

4. To differentiate $y = \tan.^{-1} x$.

$$\tan. y = x \therefore \sec.^2 y dy = dx \therefore \frac{dy}{dx} = \frac{1}{\sec.^2 y} = \frac{1}{1 + x^2}.$$

5. To differentiate $y = \cot.^{-1} x$.

$$\cot. y = x \therefore - \operatorname{cosec}.^2 y dy = dx \therefore \frac{dy}{dx} = -\frac{1}{\operatorname{cosec}.^2 y} = \frac{-1}{1 + x^2}.$$

6. To differentiate $y = \sec.^{-1} x$.

$$\sec. y = x, \therefore \tan. y \sec. y dy = dx \therefore \frac{dy}{dx} = \frac{1}{\tan. y \sec. y}$$

$$= \frac{1}{x\sqrt{x^2 - 1}}.$$

7. To differentiate $y = \operatorname{cosec}.^{-1} x$.

$$\operatorname{cosec}. y = x \therefore - \operatorname{cotan}. y \sec. y dy = dx \therefore \frac{dy}{dx} = \frac{-1}{\cot. y \sec. y} =$$

$$-\frac{1}{x\sqrt{x^2 - 1}}.$$

EXAMPLES.

1. $y = \sin.^{-1} mx \therefore \dfrac{dy}{dmx} = \dfrac{1}{\sqrt{1 - m^2 x^2}} \therefore \dfrac{dy}{dx} = \dfrac{m}{\sqrt{1 - m^2 x^2}}$

2. $y = x \sin.^{-1} x^2 \therefore \dfrac{dy}{dx} = \sin.^{-1} x^2 + x \dfrac{d \sin.^{-1} x^2}{dx}$ and

$$\frac{d \sin.^{-1} x^2}{dx} = \frac{dx}{\sqrt{1 - x^4}}.$$

$$\therefore \frac{dy}{dx} = \sin.^{-1} x^2 + \frac{2x^2}{\sqrt{1 - x^4}}.$$

3. $y = \cos.^{-1} x\sqrt{1 - x^2}$. Put $x\sqrt{1 - x^2} = z$

$$\therefore dy = \frac{-dz}{\sqrt{1 - z^2}}.$$

but $dz = (\sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}})\,dx$, and $\sqrt{1-z^2} = \sqrt{1 - x^2 + x^4}$

$$\therefore \frac{dy}{dx} = \frac{-1 + 2x^2}{\sqrt{(1 - x^2 + x^4)\,(1 - x^2)}}.$$

4. $y = \tan.^{-1} \dfrac{x}{2}$.

$$\frac{dy}{\frac{1}{2}dx} = \frac{1}{1 + (\frac{a}{2})^2} \therefore \frac{dy}{dx} = \frac{8}{4 + x^2}.$$

5. $y = \cot.^{-1}(a + mx)^2 \therefore dy = -\dfrac{1}{1+(a+mx)^4}\, d\,(a + mx)^2.$

$d\,(a + mx)^2 = 2\,(a + mx)\,mdx \therefore \dfrac{dy}{dx} = \dfrac{2m\,(a + mx)}{1+(a+mx)^4}.$

6. $y = \sec.^{-1} \dfrac{a}{x^m} \therefore dy = \dfrac{x^m}{a\sqrt{(\frac{a}{x^m})^2 - 1}}\, d\,\dfrac{a}{x^m}$, and

$$d\,\frac{a}{x^m} = -\frac{am\,dx}{x^{m+1}}.$$

$$\therefore \frac{dy}{dx} = -\frac{mx^{m-1}}{\sqrt{a^2 - x^{2m}}}.$$

7. $y = \operatorname{cosec.}^{-1} \dfrac{\sqrt{1 + x^2}}{x} \therefore dy = -d\,\dfrac{\sqrt{1 + x^2}}{x} \div$

$$\frac{\sqrt{1 + x^2}}{x}\,(\frac{1 + x^2}{x} - 1)^{\frac{1}{2}},$$

$$= - d\frac{\sqrt{1 + x^2}}{x} \div \frac{\sqrt{1 + x^2}}{x^3}, \text{ but } d\,\frac{\sqrt{1 + x^2}}{x}$$

$$= - \frac{1}{x^2\sqrt{1 + x^2}}\,dx,$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + x^2}.$$

8. $y = (\sin.^{-1} x)^2 \therefore \dfrac{dy}{dx} = 2 \sin.^{-1} x \dfrac{1}{\sqrt{1 - x^2}}.$

9. $y = \cos.^{-1} \dfrac{1}{\sqrt{1 + z^2}} \therefore \dfrac{dy}{dx} = \dfrac{1}{1 + x^2}.$

1. $y = \tan.^{-1} \sqrt{\dfrac{1 - x}{1 + x}} \therefore \dfrac{dy}{dx} = - \dfrac{1}{2\sqrt{1 - x^2}}.$

11. $y = (\cot.^{-1} x)^2 \therefore \dfrac{dy}{dx} = - \dfrac{2}{1 + x^2} \cot.^{-1} x.$

12. $y = \sec.^{-1} x^n \therefore \dfrac{dy}{dx} = \dfrac{n}{x\sqrt{x^{2n} - 1}}.$

13. $y = \operatorname{cosec}.^{-1} mx^2 \therefore \dfrac{dy}{dx} = \dfrac{2}{x\sqrt{m^2x^4 - 1}}.$

14. $y = \operatorname{versin}.^{-1} e^x \therefore \dfrac{dy}{dx} = \dfrac{e^{\frac{x}{2}}}{\sqrt{2 - e^x}}.$

(16.) In the preceding expressions the radius of the arc is always represented by unity, but, as the differentials are frequently required to radius r, we shall terminate this chapter with the several formulas in (15) accommodated to this radius. We must observe, that as y and x are homogeneous in each of those forms, $\dfrac{y}{x}$ is always a number, so that this ratio in the limit, that is $\dfrac{dy}{dx}$, is a number. Hence, r must be introduced as a multiplier so as to render the numerator and denominator of each expression of the same dimensions. The formulas, therefore, become

$$d \sin.^{-1} x = \frac{rdx}{\sqrt{r^2 - x^2}}.$$

$$d\cos.^{-1}x = -\frac{rdx}{\sqrt{r^2 - x^2}}.$$

$$d\,\text{versin.}^{-1}x = \frac{rdx}{\sqrt{2rx - x^2}}.$$

$$d\tan.^{-1}x = \frac{r^2dx}{r^2 + x^2}.$$

$$d\cot.^{-1}x = -\frac{r^2dx}{r^2 + x^2}.$$

$$d\sec.^{-1}x = \frac{r^2dx}{x\sqrt{x^2 - r^2}}.$$

$$d\,\text{cosec.}^{-1}x = -\frac{r^2dx}{x\sqrt{x^2 - r^2}}.$$

On successive Differentiation.

(17.) Since the differential coefficient derived from any function of a variable may* contain that variable, this coefficient itself may be differentiated, and we thus derive a *second differential coefficient.* In like manner, by differentiating this second coefficient, if the variable still enters it, we obtain a *third differential coefficient*, and in this way we may continue the successive differentiation till we arrive at a coefficient without the variable, when the process must terminate.

Thus, taking the function $y = ax^4$, we have, for the first differential coefficient, $\frac{dy}{dx} = 4ax^3$, as this coefficient contains x, we have, by differentiating it, the second differential coefficient $= 12ax^2$; continuing the process, we have $24ax$ for the third differential coefficient, and $24a$ for the fourth, which being constant its differential coefficient is 0.

If we were to express these several coefficients agreeably to the notation hitherto adopted, they would be

$$\text{first diff. coef. } \frac{dy}{dx} = 4ax^3,$$

$$\text{second diff. coef. } \frac{d\frac{dy}{dx}}{dx} = 12ax^2,$$

* It must contain the variable, unless in the single case of its being constant. *Ed.*

$$\text{third diff. coef. } \frac{d\,\dfrac{d\,\dfrac{dy}{dx}}{dx}}{dx} = 24\,ax,$$

&c.

But this mode of expressing the successive coefficients is obviously very inconvenient, and they are accordingly written in the following more commodious manner:

$$\text{first diff. coef. } \frac{dy}{dx}$$

$$\text{second diff. coef. } \frac{d^2y}{dx^2},$$

$$\text{third diff. coef. } \frac{d^3y}{dx^3},$$

$$\text{nth diff. coef. } \frac{d^ny}{dx^n},$$

in which notation it is to be observed, that d^2, d^3, &c. are not powers but symbols, standing in place of the words *second differential*, *third differential*, &c. The expresssions dx^2, dx^3, &c. are on the contrary powers, not, however, of x, but of dx: to distinguish the differential of a power from the power of differential, a dot is placed in the former case between d and the power.

(18.) The following are a few illustrations of the process of successive differentiation:

1. $y = x^m$.

$$\therefore \frac{dy}{dx} = mx^{m-1},$$

$$\frac{d^2y}{dx^2} = m\,(m-1)\,x^{m-2},$$

$$\frac{d^3y}{dx^3} = m\,(m-1)\,(m-2)\,x^{m-3},$$

$$\frac{d^4y}{dx^4} = m\,(m-1)\,(m-2)\,(m-3)\,x^{m-4},$$

&c. &c.

2. $u = yz$, both y and z being functions of x,

$$\therefore \frac{du}{dx} = y\frac{dz}{dx} + z\frac{dy}{dx}$$

$$\frac{d^2u}{dx^2} = y\frac{d^2z}{dx^2} + \frac{dydz}{dx^2} + \frac{zd^2y}{dx^2} + \frac{dzdy}{dx^2}$$

$$= y\frac{d^2z}{dx^2} + 2\frac{dydz}{dx^2} + \frac{zd^2y}{dx^2}$$

$$\frac{d^3u}{dx^3} = y\frac{d^3z}{dx^3} + 3\frac{dyd^2z}{dx^3} + 3\frac{dzd^2y}{dx^3} + \frac{zd^3y}{dx^3}$$

&c. &c.

3. $y = \log. x.$

$$\therefore \frac{dy}{dx} = \frac{1}{x} \qquad \frac{d^4y}{dx^4} = -\frac{2 \cdot 3}{x^4}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2} \qquad \frac{d^5y}{dx^5} = \frac{2 . 3 \cdot 4}{x^5}$$

$$\frac{d^3y}{dx^3} = \frac{2}{x^3} \qquad \frac{d^6y}{dx^6} = -\frac{2 \cdot 3 \cdot 4 \cdot 5}{x^6}$$

&c.

4. $y = e^x$.

$$\therefore \frac{dy}{dx} = e^x \qquad \frac{d^3y}{dx^3} = e^x$$

$$\frac{d^2y}{dx^2} = e^x \qquad \frac{d^4y}{dx^4} = e^x$$

&c.

If instead of e the base were a, the several coefficients would be

$$\log. a \cdot a^x,\ \log.^2 a \cdot a^x,\ \log.^3 a \,.\, a^x,\ \log.^4 a \cdot a^x,\ \&c.$$

It appears, therefore, that exponental functions possess this property, viz. that $\frac{d^ny}{dx^n} \div y$ is always constant.

5. $y = \sin. x.$

$$\therefore \frac{dy}{dx} = \cos. x \qquad \frac{d^3y}{dx^3} = -\cos. x$$

$$\frac{d^2y}{dx^2} = -\sin. x \qquad \frac{d^4y}{dx^4} = \sin. x$$

&c.

We need not multiply examples here, as the process of successive differentiation will be very frequently employed in the next two chapters.

CHAPTER III.

ON MACLAURIN'S THEOREM.

(19.) If y represent a function of x, which it is possible to develop in a series of positive ascending powers of that variable, then will that development be

$$y = [y] + \left[\frac{dy}{dx}\right] x + \frac{1}{2}\left[\frac{d^2y}{dx^2}\right] x^2 + \frac{1}{2.3}\left[\frac{d^3y}{dx^3}\right] x^3 + \&c.$$

where the brackets are intended to intimate that the functions which they enclose are to be taken in that particular state, arising from taking $x = 0$.*

For, since by hypothesis

$$\begin{aligned}
y &= A + Bx + Cx^2 + Dx^3 + Ex^4 + \&c. \ldots (1) \\
\therefore \frac{dy}{dx} &= B + 2Cx + 3Dx^2 + 4Ex^3 + \&c. \\
\frac{d^2y}{dx^2} &= 2C + 2 \cdot 3Dx + 3 . 4Ex^2 + \&c. \\
\frac{d^3y}{dx^3} &= 2 . 3D + 2 . 3 . 4Ex + \&c. \\
\&c. & \qquad \&c.
\end{aligned}$$

Let, now, $x = 0$, then

$$\begin{aligned}
[y] &= A \\
\left[\frac{dy}{dx}\right] &= B \\
\left[\frac{d^2y}{dx^2}\right] &= 2C \therefore C = \frac{1}{2}\left[\frac{d^2y}{dx^2}\right] \\
\left[\frac{d^3y}{dx^3}\right] &= 2.3\,D \therefore D = \frac{1}{2 \cdot 3}\left[\frac{d^3y}{dx^3}\right] \\
\&c. & \qquad \&c.
\end{aligned}$$

* This plan of enclosing the differential coefficient in brackets we shall usually adopt, when we wish to express not the *general* state of this function, but that state which arises from the variable taking a *particular* value. What that value is will generally be made known by the nature of the inquiry.

Hence, by substitution, equation (1) becomes

$$y=[y]+[\frac{dy}{dx}]\,x+\frac{1}{2}[\frac{d^2y}{dx^2}]x^2+\frac{1}{2\cdot 3}[\frac{d^3y}{dx^3}]\,x^3+\&c. \ldots . (2),$$

which is Maclaurin's theorem for the development of a function, according to the ascending powers of the variable. We shall apply it to some examples.

EXAMPLES.

(20.) 1. Let it be required to develop $(a+x)^n$, the exponent n being any uumber whatever, either positive or negative, whole or fractional, rational or irrational.

Put $y=(a+x)^n$ . . therefore . $[y]=a^n$

$$\therefore \frac{dy}{dx}=n(a+x)^{n-1} \ldots\ldots [\frac{dy}{dx}]=na^{n-1}$$

$$\frac{d^2y}{dx^2}=n(n-1)\,(a+x)^{n-2} \ldots [\frac{d^2y}{dx^2}]=n(n-1)a^{n-2}$$

$$\frac{d^3y}{dx^3}=n(n-1)\,(n-2)\,(a+x)^{n-3},\ [\frac{d^3y}{dx^3}]=$$

$$n(n-1)\,(n-2)a^{n-3}$$

&c. &c.

Substituting these values for the coefficients in the foregoing theorem, there results

$$(a+x)^n=a^n+na^{n-1}\,x+\frac{n\,(n-1)}{2}\,a^{n-2}\,x^2+\frac{n\,(n-1)(n-2)}{2\cdot 3}$$
$$a^{n-3}\,x^3+\&c.$$

and thus the truth of the *Binomial Theorem* is established in its utmost generality.

2. To develop log. $(a+x)$.

Put $y=\log.\,(a+x)$, therefore $[y]=\log.\,a$

$$\frac{dy}{dx}=\frac{1}{a+x} \ldots . [\frac{dy}{dx}]=\log.\,\frac{1}{a}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{(a+x)^2} \ldots [\frac{d^2y}{dx^2}]=-\frac{1}{a^2}$$

$$\frac{d^3y}{dx^3}=\frac{2}{(a+x)^3} \ldots [\frac{d^3y}{dx^3}]=\frac{2}{a^3}$$

$$\frac{d^4y}{dx^4}=-\frac{2\cdot 3}{(a+x)^4} \ldots [\frac{d^4y}{dx^4}]=-\frac{2.3}{a^4}$$

&c. &c.

$$\therefore \log. (a + x) = \log. a + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \&c.$$

If $y = \log. x$ were proposed, then, since $[y]$, $[\frac{dy}{dx}]$, &c. are infinite, we infer, for reasons similar to those assigned at art. 4, that the development in the proposed form is impossible.

3. To develop $\sin. x$.

$$\begin{array}{llll}
y & = \sin. x & \ldots\ldots & [y] = 0 \\
\frac{dy}{dx} & = \cos. x & \ldots\ldots & [\frac{dy}{dx}] = 1 \\
\frac{d^2y}{dx^2} & = -\sin. x & \ldots\ldots & [\frac{dy^2}{dx^2}] = 0 \\
\frac{d^3y}{dx^3} & = -\cos. x & \ldots\ldots & [\frac{d^3y}{dx^3}] = -1 \\
\frac{d^4y}{dx^4} & = \sin. x & \ldots\ldots & [\frac{d^4y}{dx^4}] = 0 \\
\frac{d^5y}{dx^5} & = \cos. x & \ldots\ldots & [\frac{d^5y}{dx^5}] = 1 \\
\&c. & & & \&c.
\end{array}$$

$$\therefore \sin. x = x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2 . 3\cdot 4\cdot 5} - \&c.$$

4. To develop $\cos. x$.

$$\begin{array}{llll}
y & = \cos. x & \ldots\ldots & [y] = 1 \\
\frac{dy}{dx} & = -\sin. x & \ldots\ldots & [\frac{dy}{dx}] = 0 \\
\frac{d^2y}{dx^2} & = -\cos. x & \ldots\ldots & [\frac{d^2y}{dx^2}] = -1 \\
\&c. & & & \&c.
\end{array}$$

$$\therefore \cos. x = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \&c.$$

5. To develop a^x.

$$\begin{array}{llll}
y & = a^x & \text{therefore} & [y] = 1 \\
\frac{dy}{dx} & = {}^*Aa^x & \ldots\ldots & [\frac{dy}{dx}] = A
\end{array}$$

* A is put here for the hyperbolic logarithm of the base a, that is, for the expression

$$(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \&c.$$

$$\frac{d^2y}{dx^2} = A^2a^x \quad . \quad . \quad . \quad . \quad [\frac{d^2y}{dx^2}] = A^2$$

$$\frac{d^3y}{dx^3} = A^3a^x \quad . \quad . \quad . \quad . \quad [\frac{d^3y}{dx^3}] = A^3$$

&c. &c.

$$\therefore a^x = 1 + Ax + \frac{A^2x^2}{1 \cdot 2} + \frac{A^3x^3}{1 \cdot 2 \cdot 3} + \&c.$$

which is the *Exponential Theorem.*

Since $A = \log. a$, we may give to the development the form

$$a^x = 1 + x \log. a + \frac{1}{2} (x \log. a)^3 + \frac{1}{2.3} (x \log. a)^3 + \&c.$$

For $x = 1$, we have the following expression for any number, a, in terms of its Napierian logarithm:

$$a = 1 + \log. a + \frac{1}{2} \log.^2 a + \frac{1}{2 \,.\, 3} \log.^3 a + \&c.$$

changing a into the Napierian base, e, we have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \,.\, 3} + \&c.$$

which, when $x = 1$, gives, for the base e, the value

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{2 \,.\, 3} + \&c.$$

(21.) From the development of e^x may be immediately derived several very curious and useful analytical formulas, and we shall avail ourselves of this opportunity to present the principal ones to the notice of the student.

If, in the development of e^x, we put $z\sqrt{-1}$ for x, we shall have

$$e^{z\sqrt{-1}} = 1 + z\sqrt{-1} - \frac{z^2}{1 \cdot 2} - \frac{z^3\sqrt{-1}}{1 \,.\, 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$$

and, changing the sign of the radical,

$$e^{-z\sqrt{-1}} = 1 - z\sqrt{-1} - \frac{z^2}{1 \cdot 2} + \frac{z^3\sqrt{-1}}{1 \cdot 2 \cdot 3} + \frac{z^4}{1 \cdot 2 \cdot 3 \cdot 4} - \&c.$$

If these expressions be first added and then subtracted, there will result the following remarkable developments, viz.

$$\frac{e^{z\sqrt{-1}} + e^{z\sqrt{-1}}}{2} = 1 - \frac{z^2}{1 \cdot 2} + \frac{z^1}{1 \cdot 2 \cdot 3 \cdot 4} - \&c.$$

$$\frac{e^{z\sqrt{-1}} - e^{z\sqrt{-1}}}{2\sqrt{-1}} = z - \frac{z^3}{1 \cdot 2 \cdot 3} + \frac{z^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

Now it has been seen (examples 4 and 3) that these two series are also the respective developments of cos. z and sin. z; hence, putting x instead of z, we may conclude that

$$\sin. x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}} \quad \ldots . (1)$$

$$\cos. x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2} \quad \ldots . (2)$$

where the sine and cosine of a *real* arc are expressed by imaginary exponentials.

These expressions were first deduced by Euler, and are considered by Lagrange as among the finest analytical discoveries of the age. (*Calcul des Fonctions*, page 114.)

(22.) If for the real arc x we substitute the imaginary arc $x\sqrt{-1}$ we shall have

$$\sin. (x\sqrt{-1}) = \frac{e^{-x} - e^{x}}{2\sqrt{-1}} \quad \ldots . (3)$$

$$\cos. (x\sqrt{-1}) = \frac{e^{-x} + e^{x}}{2} \quad \ldots . (4)$$

Also, since $\frac{\sin.}{\cos.} = \tan.$, it follows, from (1) and (2), that

$$\sqrt{-1} \tan. x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}} = \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1}*$$

By multiplying equation (1) by $\pm \sqrt{-1}$, and adding the result to (2), we have

$$\cos. x \pm \sin. x \sqrt{-1} = e^{\pm x\sqrt{-1}} \quad \ldots . (5;)$$

or if we change x into mx,

$$\cos. mx \pm \sin. mx \sqrt{-1} = e^{\pm mx\sqrt{-1}} \quad \ldots . (6),$$

but $e^{\pm mx\sqrt{-1}}$ is $e^{\pm x\sqrt{-1}}$ raised to the mth power. Hence this singular property, viz.

* Multiplying the numerator and denominator of the second member of the equation by $e^{x\sqrt{-1}}$. *Ed.*

$$(\cos. x \pm \sin. x \sqrt{-1})^m = \cos. mx \pm \sin. mx \sqrt{-1} \; \ldots . \; (7),$$

which was discovered by de Moivre, and is hence called *De Moivre's formula.*

If the first side of this equation be developed by the binomial theorem, it becomes

$$\cos.^m x \pm m \cos.^{m-1} xp + \frac{m(m-1)}{2} \cos.^{m-2} xp^2 \pm \&c.$$

p being put for the imaginary $\sqrt{-1} \sin. x$.

Now in any equation, the imaginaries on one side are equal to those on the other, (*Algebra*); hence, expunging from this expression all the imaginaries, that is, all the terms containing the odd powers of p, we have, in virtue of (7),

$$\cos. mx = \cos.^m x - \frac{m(m-1)}{2} \cos.^{m-2} x \sin.^2 x +$$
$$\frac{m(m-1)(m-2)(m-3}{2 \cdot 3 \cdot 4} \cos.^{m-4} x \sin.^4 x - \&c.$$

In like manner, equating $\sin. mx \sqrt{-1}$ with the imaginary part of the above development, and then dividing by $\sqrt{-1}$, we have

$$\sin. mx = m \cos.^{m-1} x \sin. x - \frac{m(m-1)(m-2)}{2 \cdot 3} \cos.^{m-3} x \sin.^3 x + \&c.$$

From these two series the sine and cosine of a multiple arc may be determined from the sine and cosine of the arc itself.

(28.) If in the formula (2) we represent $e^{x\sqrt{-1}}$ by y, then $e^{-x\sqrt{-1}} = \frac{1}{y}$; therefore,

$$2 \cos. x = y + \frac{1}{y}$$

or, if in the same formula mx be put for x, we have

$$2 \cos. mx = y^m + \frac{1}{y^m}$$

and from these two equations we deduce the following, viz.

$$y^2 - 2y \cos. x + 1 = 0 \; \ldots . \; (1)$$
$$y^{2m} - 2y^m \cos. mx + 1 = 0 \; \ldots . \; (2).$$

Since these equations exist simultaneously, the latter must have two of its roots or values of y equal to the two roots of the former,

and must, therefore, be divisible by it; or, putting θ for mx, we have

$$y^{2m} - 2y^m \cos. \theta + 1 = 0 \quad . \quad . \quad . \quad . \quad (3),$$

divisible by

$$y^2 - 2y \cos. \frac{\theta}{m} + 1 = 0 \quad . \quad . \quad . \quad . \quad (4).$$

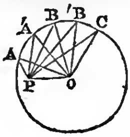
But $\cos. \theta = \cos. (\theta + 2n\pi)$, n being any whole number, and $\pi = 180°$; hence, making successively $n = 0, = 1, = 2$, &c, to $n = m - 1$, we have, since the first equation continues to be divisible by the second in these cases,

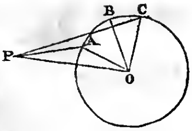
$$y^{2m} - 2y^m \cos. \theta + 1 = (y^2 - 2y \cos. \frac{\theta}{m} + 1)$$
$$\times (y^2 - 2y \cos. \frac{\theta + 2\pi}{m} + 1)$$
$$\times (y^2 - 2y \cos. \frac{\theta + 4\pi}{m} + 1)$$
$$\times (y^2 - 2y \cos. \frac{\theta + 6\pi}{m} + 1) \text{ \&c. to } m \text{ factors.}$$

The truth of this equation is obvious, for, while the substitution of $\theta + 2n\pi$ for θ causes no alteration in the expression (3), the same substitution in (4) gives to that expression a new value, for every value of n, from $n = 0$ to $n = m - 1$, for the arcs $\frac{\theta}{m}, \frac{\theta + 2\pi}{m}$ &c. are all different. As, therefore, the expression (3) is divisible by (4) under all these m changes of value, it is plain that these are its m quadratic factors.

In this way may any trinomial of the form $y^{2m} - 2ky^m + 1$ be decomposed into its quadratic factors, provided k does not exceed unity, for then k may always be replaced by the cosine of an arc.

(24.) The geometrical interpretation of the foregoing equation, presents a curious property of the circle, first discovered by De Moivre. To exhibit this property, let P be any point either within or without the circle whose centre is O, and let the circumference be divided into any number of equal parts, commencing at any point A. Join the points of division, A, B, C, &c. to P, then, since in the fore-





going analytical expression the radius OA is expressed by unity, we shall have, by introducing the radius itself so as to render the terms homogeneous, the following geometrical values of the above factors, where it is to be observed that

$$\angle \text{POA} = \frac{\theta}{m} \text{ and OP} = y,$$

$$y^{2m} - 2y^m \cos. \theta + 1 = \text{OP}^{2m} - 2\text{OP}^m \times \text{OA}^m \cos. m\,(\text{AOP}) + \text{AO}^{2m}$$

$$y^2 - 2y \cos. \frac{\theta}{m} + 1 = \text{OP}^2 - 2\text{OP} \times \text{OA} \cos. \text{AOP} + \text{AO}^2 = \text{PA}^2 *$$

$$y^2 - 2y \cos. \frac{\theta + 2\pi}{m} + 1 = \text{OP}^2 - 2\text{OP} \times \text{OA} \cos. \text{BOP} + \text{BO}^2 = \text{PB}^2$$

$$y^2 - 2y \cos. \frac{\theta + 4\pi}{m} + 1 = \text{OP}^2 - 2\text{OP} \times \text{OA} \cos. \text{COP} + \text{CO}^2 = \text{PC}^2$$

&c. &c. &c.

Hence,

$$\text{OP}^{2m} - 2\text{OP}^m \times \text{OA}^m \cos. m\,(\text{AOP}) + \text{OA}^{2m} = \text{PA}^2 \times \text{PB}^2 \times \text{PC}^2 \times \&c.$$

and this is *Demoivre's property of the circle.*

(25.) If AOP = 0, that is, if P be upon the radius through one of the points of division A, then cos. m (AOP) = 1. Hence,

$$\text{OP}^{2m} - 2\text{OP}^m \times \text{OA}^m + \text{OA}^{2m} = \text{PA}^2 \times \text{PB}^2 \times \text{PC}^2 \times \&c.$$

consequently, extracting the square root of each member,

$$\text{OP}^m \backsim \text{OA}^m = \text{PA} \times \text{PB} \times \text{PC} \times \&c.$$

If the arcs AB, BC, &c. be bisected by A′, B′, &c. the circumference will be divided into $2m$ equal parts, and, by the equation just deduced

$$\text{OP}^{2m} \backsim \text{OA}^{2m} = \text{PA} \times \text{PA}' \times \text{PB} \times \text{PB}' \times \&c.$$

that is

$$\text{OP}^{2m} \backsim \text{OA}^{2m} = (\text{OP}^m \backsim \text{OA}^m)\ \text{PA}' \times \text{PB}' \times \&c.$$

therefore,

$$\frac{\text{OP}^{2m} \backsim \text{OA}^{2m}}{\text{OP}^m \backsim \text{OA}^m} = \text{OP}^m + \text{OA}^m = \text{PA}' \times \text{PB}' \times \text{PC}' \times \&c.$$

and these are *Cotes's* properties of the circle.

(26.) If now we return to the expression (5), and suppose $x = \frac{\pi}{2}$, it becomes

* Gregory's Trigonometry, p. 54, or Lacroix's Trigonometry.

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$$\sqrt{-1} = e^{\frac{\pi}{2}\sqrt{-1}} \therefore \log.\sqrt{-1} = \frac{\pi}{2}\sqrt{-1},$$

and

$$(\sqrt{-1})^{\sqrt{-1}} = e^{-\frac{\pi}{2}}. = \frac{1}{e^{\frac{\pi}{2}}} = \frac{1}{(2.71828)^{1.5708}}$$

From the second of these equations we get

$$\pi = \frac{2\log.\sqrt{-1}}{\sqrt{-1}} = \frac{\log.(\sqrt{-1})^2}{\sqrt{-1}} = \frac{\log.-1}{\sqrt{-1}} = -\sqrt{-1}$$

$\log. - 1.$

From the third

$$(\sqrt{-1})^{\sqrt{-1}} = 1 - \frac{\pi}{2} + \frac{1}{1\cdot 2}\cdot\frac{\pi^2}{4} - \frac{1}{1\cdot 2\cdot 3}\cdot\frac{\pi^3}{8} + \&c.$$

Two very singular results; first obtained by *John Bernouilli.*

6. To develop tan. x.

$$\begin{array}{llll}
y & = \tan. x, \text{ therefore} & \ldots & [y] = 0 \\
\dfrac{dy}{dx} & = \sec.^2 x & \ldots\ldots\ldots & [\dfrac{dy}{dx}] = 1 \\
\dfrac{d^2y}{dx^2} & = 2\sec.^2 x \tan. x & \ldots & [\dfrac{d^2y}{dx^2}] = 0 \\
\dfrac{d^3y}{dx^3} & = 2\sec.^2 x\,(1 + 3\tan.^2 x) & . & [\dfrac{d^3y}{dx^3}] = 2
\end{array}$$

We thus see that the first two terms of the development are $x + \frac{2x^3}{1\cdot 2\cdot 3}$, but we shall not continue the differentiation, since it does not make known the law of the series. The development will be more readily obtained by means of those already given for sin. x and cos. x, as follows:

$$\tan. x = \frac{x - \dfrac{x^3}{1\cdot 2\cdot 3} + \dfrac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \&c.}{1 - \dfrac{x^2}{2} + \dfrac{x^4}{1\cdot 2\cdot 3\cdot 4} - \&c.}$$

therefore the development, found by actual division, is

$$\tan. x = x - \frac{2x^3}{1\cdot 2\cdot 3} + \frac{16x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \&c.$$

but, to obtain the *law* of the terms and thus be enabled to continue them at pleasure, it will be best to apply the method of indeterminate coefficients. Assume, therefore, this fraction equal to the series

$$A_1\,x + A_3\,x^3 + A_5\,x^5 + A_7\,x^7 + \&c.$$

Multiplying this by the denominator, we have this expression for the numerator, viz.

$$x - \frac{x^3}{1\cdot2\cdot3} + \frac{x^5}{1.2\cdot3\cdot4\cdot5} - \&c =$$

$$A_1\, x + \left.\begin{array}{l} A_3 \\ -\dfrac{A_1}{1\cdot2} \end{array}\right| x^3 + \left.\begin{array}{l} A_5 \\ -\dfrac{A_3}{1\cdot2} \\ +\dfrac{A_1}{1\cdot2\cdot3\cdot4} \end{array}\right| x^5 + \&c.$$

Hence, equating the coefficients of the like terms, there results

$$A_1 = 1 \qquad \text{therefore} \qquad A_1 = 1$$

$$A_3 - \frac{A_1}{1\cdot2} = -\frac{1}{1\cdot2\cdot3} \quad \ldots\ldots \quad A_3 = \frac{A_1}{1\cdot2} - \frac{1}{1\cdot2\cdot3}$$

$$= \frac{2}{1\cdot2\cdot3}$$

$$A_5 - \frac{A_3}{1\cdot2} + \frac{A_1}{1\cdot2\cdot3\cdot4} = \frac{1}{1\cdot2\cdot3\cdot4\cdot5},\ A_5 = \frac{A_3}{1\cdot2} - \frac{A_1}{1\cdot2\cdot3\cdot4}$$

$$+ \frac{1}{1\cdot2\cdot3\cdot4\cdot5}$$

&c. &c.

the law of these coefficients being such, that

$$A_{2n+1} = \frac{A_{2n-1}}{1\cdot2} - \frac{A_{2n-3}}{1\cdot2\cdot3\cdot4} + \ldots \pm \frac{A_1}{1\cdot2.3\ldots2n} \ldots \pm$$

$$\frac{1}{1\cdot2\cdot3\,..\,(2n+1)}.$$

7. To develop $\tan.^{-1} x$.

$$y = \tan.^{-1} x \quad \ldots\ldots \text{therefore} \ldots\ldots \quad [y] = 0$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} \quad \ldots\ldots\ldots\ldots \quad \left[\frac{dy}{dx}\right] = 1$$

$$\frac{d^2y}{dx^2} = -\frac{2x}{(1+x^2)^2} \quad \ldots\ldots\ldots\ldots \quad \left[\frac{d^2y}{dx^2}\right] = 0$$

$$\frac{d^3y}{dx^3} = -\frac{2x}{(1+x^2)^2} + \frac{2\cdot4x^2}{(1+x^2)^3} \quad \ldots\ldots \quad \left[\frac{d^3y}{dx^3}\right] = -2$$

$$\frac{d^4y}{dx^4} = \frac{2^3x}{(1+x^2)^3} + \frac{2^4x}{(1+x^2)^3} - \frac{2^4\cdot3x^3}{(1+x^2)^4} \ldots \left[\frac{d^4y}{dx^4}\right] = 0$$

$\frac{d^5y}{dx^5} = \frac{2^3 \cdot 3}{(1 + x^2)^3} - \frac{2^5 . 3^2x^2}{(1 + x^2)^4} + \frac{2^7 \cdot 3\, x^4}{(1 + x^2)^5}$. $\cdot\ [\frac{d^5y}{dx^5}] = 2^3 . 3$

&c. &c.

$$\therefore y = \tan. y - \tfrac{1}{3} \tan.{}^3 y + \tfrac{1}{5} \tan.{}^5 y - \tfrac{1}{7} \tan.{}^7 y + \&c.$$

If $y = 45°$, then $\tan. y = 1$;

$$\therefore \text{arc } 45° = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \&c.$$

(27.) From this series an approximation may be made to the circumference of a circle, but, from its very slow convergency, it is not eligible for this purpose. *Euler* has obtained from the above general development a series much more suitable, by help of the known formula, (*Gregory's Trig.*, page 46,)

$$\tan. (a + b) = \frac{\tan. a + \tan. b}{1 - \tan. a \tan. b}*$$

for, when $a + b = 45°$, $\tan. (a + b) = 1$; therefore,

$$\tan. a + \tan. b = 1 - \tan. a \tan. b.$$

If either tan. a or tan. b were given, the other would be determinable from this equation. Thus, if we suppose,

$$\tan. a = \frac{1}{n}, \text{ then } \frac{1}{n} + \tan. b = 1 - \frac{\tan. b}{n}, \therefore \tan. b = \frac{n - 1}{n + 1}.$$

Now the value of n is arbitrary, and our object is to assume it so that the sum of the series, expressing the arcs a, b, in terms of their tangents, may be the most convergent. This value appears to be $n = 2$, or $n = 3$; therefore, taking $n = 2$, we have

$$\tan. a = \tfrac{1}{2}, \tan. b = \tfrac{1}{3}.$$

Hence, substituting in the general development a for y and $\frac{1}{2}$ for tan. y, and then again b for y and $\frac{1}{3}$ for tan. b, the sum of the resulting series will express the length of the arc $a + b = 45°$, that is

$$\text{arc. } 45° = \frac{1}{2} - \frac{1}{3 . 2^3} + \frac{1}{5 . 2^5} - \frac{1}{7 . 2^7} + \&c.$$
$$+ \frac{1}{2} - \frac{1}{3 . 3^3} + \frac{1}{5 . 3^5} - \frac{1}{7 . 3^7} + \&c.$$

(28.) Another form of development, still more convergent than this, has been obtained by *M. Bertrand* from the formula

$$\tan. 2a = \frac{2 \tan. a}{1 - \tan.{}^2 a}$$

* Lacroix Trigonometry.

For put tan. $a = \frac{1}{5}$, then tan. $2a = \frac{5}{12}$, therefore $2a \angle 45°$, because tan. $45° = 1$: from this value of $2a$ we deduce

$$\text{tan. } 4a = \frac{2 \text{ tan. } 2a}{1 - \text{tan.}^2 2a} = \frac{120}{119}$$

$$\therefore 4a \,7\, 45°.$$

Let now $4a = A$, $45° = B$, $A - B = b =$ excess of $4a$ above $45°$, then we have $45° = A - b$. But

$$\text{tan. } (A - B) = \text{tan. } b = \frac{\text{tan. } A - \text{tan. } B}{1 + \text{tan. } A \text{ tan. } B} = \frac{1}{239}$$

Consequently, if in the general development we replace y by a and tan. y by $\frac{1}{5}$, and then multiply by 4, we shall have the length of the arc $4a$, and, since this arc exceeds $45°$ by the arc b, if we subtract the development of this latter, which is given by substituting $\frac{1}{239}$ for tan. y, the remainder will be the true development of $45°$. Thus

$$45° = 4\left(\frac{1}{2} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \frac{1}{7 \cdot 5^7} + \&c.\right)$$
$$- \left(\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \&c.\right)$$

This series is very convergent, and, by taking about 8 terms in the first row and 3 in the second, we find, for the length of the semicircle, the following value, viz.

$$\pi = 3 \cdot 14159265358979 3.$$

If we take but three terms of the first and only one of the second, we shall have $\pi = 3 \cdot 1416$, the approximation usually employed in practice.

(29.) The following examples are subjoined for the exercise of the student:

8. To develop $y = \text{sin.}^{-1} x$.

$$y = \text{sin. } y + \frac{\text{sin.}^3 y}{1 \cdot 2 \cdot 3} + \frac{3^2 \text{ sin.}^5 y}{1 \cdot 2 \cdot 3 \cdot 4 . 5} + \frac{3^2 \cdot 5^2 \text{ sin.}^7 y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}.$$

9. To develop $y = \text{cos. }^{-1} x$.

$$y = \tfrac{1}{2} \pi - \text{cos. } y - \frac{\text{cos.}^3 y}{1 \cdot 2 \cdot 3} - \frac{3^2 \text{ cos.}^5 y}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c.$$

10. To develop $y = \text{cot. } x$ by the method of indeterminate coefficients, as in example 6.

$$\cot. x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{3^2 \cdot 5} - \frac{2x^5}{3^2 \cdot 5 \cdot 7} - \&c.$$

11. To develop $y = (a + bx + cx^2 + \&c.)^n$.

$$(a + bx + cx^2 + \&c.)^n =$$

$$a^n + na^{n-1}bx + \left.\begin{array}{r} \frac{n(n-1)}{2}a^{n-1}b^2 \\ + na^{n-1}c \end{array}\right| x^2 + \left.\begin{array}{r} \frac{n(n-1)(n-2)}{2 \cdot 3}a^{n-3}b^3 \\ + n(n-1)\,{}^{n-2}bc \\ + na^{n-1}cd \end{array}\right| x^3 + \&c.$$

This is the multinomial theorem of De Moivre. It is given in a very convenient practical form in my Treatise on Algebra.

CHAPTER IV.

ON TAYLOR'S THEOREM, AND ON THE DIFFERENTIATION AND DEVELOPMENT OF IMPLICIT FUNCTIONS.

(30.) In the second chapter we established the *form* of the general development of the function $F(x + h)$. We here propose to investigate Taylor's theorem, which is an expression exhibiting the *actual* development of the same function. The following lemma, must, however, be premised, viz. that if in any function of $p + q$ one of the quantities p, q, is variable, and the other constant, we may determine the several differential coefficients, without inquiring which is the constant and which the variable, for these coefficients will be the same, whichever be variable. This principle is almost axiomatic. For as the function contains but *one* variable we may put $p + q = x$ or $F(p + q) = Fx$, and whichever of the parts p, q, takes the increment h, the result $F(x + h)$ is necessarily the same; hence the development of this function is the same on either hypothesis, and therefore the second term of that development, and hence also the differential coefficient. The first differential coefficient being the same, the succeeding must be the same; therefore generally

$$\frac{d^nF(p + q)}{dp^n} = \frac{d^nF(p + q)}{dq^n}$$

whatever be the value of n.

Let now $y = \mathrm{F}x$, and $\mathrm{Y} = \mathrm{F}\,(x + h)$, and assume, agreeably to art. (4)

$$\mathrm{Y} = y + \mathrm{A}h + \mathrm{B}h^2 + \mathrm{C}h^3 + \&c.$$

A, B, C, &c. being unknown functions of x, which it is now required to determine.

Suppose, first, h to be variable and x constant, then, differentiating on that supposition, we have

$$\frac{d\mathrm{Y}}{dh} = \mathrm{A} + 2\mathrm{B}h + 3\mathrm{C}h^2 + \&c.$$

Suppose, secondly, that x is variable and h constant, then the differential coefficient is

$$\frac{d\mathrm{Y}}{dx} = \frac{dy}{dx} + \frac{d\mathrm{A}}{dx}h + \frac{d\mathrm{B}}{dx}h^2 + \frac{d\mathrm{C}}{dx}h^3 + \&c.$$

But by the lemma these two differential coefficients are identical, hence equating the coefficients of the like powers of h there results

$$\mathrm{A} = \frac{dy}{dx},\ \mathrm{B} = \frac{d\mathrm{A}}{2dx},\ \mathrm{C} = \frac{d\mathrm{B}}{3dx},\ \&c.$$

that is

$$\mathrm{A} = \frac{dy}{dx},\ \mathrm{B} = \frac{d^2y}{dx^2}\cdot\frac{1}{2},\ \mathrm{C} = \frac{d^3y}{dx^3}\cdot\frac{1}{2\cdot 3},\ \&c.$$

Hence the required development is

$$\mathrm{Y} = y + \frac{dy}{dx}\cdot\frac{h}{1} + \frac{d^2y}{dx^2}\cdot\frac{h^2}{1\cdot 2} + \frac{d^3y}{dx^3}\cdot\frac{h^3}{1\cdot 2\cdot 3} + \&c.$$

If h is negative, the signs of the alternate terms will be negative.

When we wish for the development of the function $\mathrm{Y} = \mathrm{F}(x+h)$ in any particular state, that is when x takes a given value, we have only to substitute this value for x in the general expressions for the coefficients previously determined, and we shall have the development according to the above form, that is, provided of course, that the development in such form is possible. But if the value chosen for x render the development impossible, the impossibility will be intimated to us from the circumstance of some of the terms becoming infinite, as explained in *art.* (4). It may, however, be proper here to remark, that even in these cases of impossibility, the leading terms of the development as given by Taylor's theorem, are still true as far as the first term that becomes infinite. But as we propose to devote

hereafter an entire chapter to the examination of the *failing cases* of Taylor's theorem, we shall not enter into the inquiry here.

(31.) The theorem of Maclaurin may be easily deduced from that of Taylor, thus:

Let x take the particular value $x = 0$, then

$$[\mathrm{Y}] = \mathrm{F}h = [\mathrm{F}x] + [\frac{d\mathrm{F}x}{dx}]\frac{h}{1} + [\frac{d^2\mathrm{F}x}{dx^2}]\frac{h^2}{1\cdot 2} + [\frac{d^3\mathrm{F}x}{dx^3}]\frac{h^3}{1\cdot 2\cdot 3} + \&c.$$

Now each of these coefficients is constant, and therefore independent of the value of h, hence h may take any value whatever, without affecting these coefficients; we may therefore call it x, it being observed that although x appears in the *notation* of the coefficients, it does not appear in the coefficients themselves. It follows, therefore, that

$$\mathrm{F}x = [\mathrm{F}x] + [\frac{d\mathrm{F}x}{dx}]\,x + [\frac{d^2\mathrm{F}x}{dx^2}]\frac{x^2}{1\cdot 2} + [\frac{d^3\mathrm{F}x}{dx^3}]\frac{x^3}{1\cdot 2\cdot 3} + \&c.$$

which is Maclaurin's theorem, before investigated.

EXAMPLES.

(32.) 1. To develop sin. $(x + h)$ in a series of powers of the arc h. Let $y = \sin. x \therefore \frac{dy}{dx} = \cos. x, \frac{d^2y}{dx^2} = -\sin. x, \frac{d^3y}{dx^3} = -\cos. x$, &c. hence, by Taylor's theorem,

$$\sin.(x+h) = \sin. x + \cos. x\, h - \sin. x\frac{h^2}{1\cdot 2} - \cos. x\frac{h^3}{1\cdot 2\cdot 3} + \&c.$$

$$= \sin. x\,(1 - \frac{h^2}{1\cdot 2} + \frac{h^4}{1\cdot 2\cdot 3.4} - \&c.)$$

$$+ \cos. x\,(h - \frac{h^3}{1\cdot 2\cdot 3} + \frac{h^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \&c.)$$

The series within the parentheses are respectively equal to cos. h and sin. h (p. 30,) hence the property

$$\sin.(x+h) = \sin. x\cos. h + \sin. h\cos. x.$$

2. To develop cos. $(x + h)$.

$$y = \cos. x \therefore \frac{dy}{dx} = -\sin. x, \frac{d^2y}{dx^2} = -\cos. x, \frac{d^3y}{dx^3} = \sin. x, \&c.$$

Hence

$$\cos.(x+h) = \cos. x - \sin. x\, h - \cos. x \frac{h^2}{1\cdot 2} + \sin. x \frac{h^3}{1\cdot 2\cdot 3} + \&c.$$

$$= \cos. x\left(1 - \frac{h^2}{1.2} - \frac{h^4}{1\cdot 2\cdot 3\cdot 4} - \&c.\right)$$

$$- \sin. x\left(h - \frac{h^3}{1\cdot 2\cdot 3} + \frac{h^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \&c.\right.$$

Hence the property*

$$\cos.(x+h) = \cos. x \cos. h - \sin. x \sin. h.$$

From this property, and the analogous one for the sine of $x + h$ deduced in last Example, the whole theory of trigonometry flows. By putting $h = (m-1)\,x$, these two properties become

$$\sin. mx = \sin. x \cos.(m-1)\,x + \sin.(m-1)\,x \cos. x.$$

$$\cos. mx = \cos. x \cos.(m-1)\,x - \sin. x \sin (m-1)\,x.$$

Two equations which will be employed to abridge the expressions for the differential coefficients in the next Example.

3. To develop $\tan.^{-1}(x+h)$.

$$y = \tan.^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{\sec.^2 y} = \cos.^2 y.$$

$$\frac{d^2y}{dx^2} = -2 \sin. y \cos. y \frac{dy}{dx} = -\sin. 2y \cos.^2 y.$$

$$\frac{d^3y}{dx^3} = -2\,(\cos. 2y \cos.^2 y - \sin. 2y \sin. y \cos. y)\frac{dy}{dx}$$

$$= -\cos. 3y \cos. y \frac{dy}{dx}$$

$$= -\cos. 3y \cos.^2 y$$

$$\frac{d^4y}{dx^4} = 2\cdot 3\,(\sin. 3y \cos.^3 y + \cos. 3y \sin. y \cos.^2 y)\frac{dy}{dx}$$

$$= 2\cdot 3 \sin. 4y \cos.^2 y \frac{dy}{dx} = 2\cdot 3 \sin. 4y \cos.^4 y$$

* It should not be concealed from the student that the property here deduced is, in fact, involved in that which we have employed to obtain the differential of a sine. If, however, we consider this differential deduced as in Note A at the end, then, the inference above, fairly establishes the property in question.

$$\frac{d^5y}{dx^5} = 2 \cdot 3 \cdot 4 \ (\cos. 4y \cos. ^4y - \sin. 4y \sin. y \cos. ^3y) \frac{dy}{dx}$$

$$= 2 \cdot 3 \cdot 4 \cos. 5y \cos. ^3y \frac{dy}{dx} = 2 \cdot 3 \cdot 4 \cos. 5y \cos. ^5y$$

&c. &c.

hence

$$\tan.^{-1} (x + h) = y + \cos. ^2y\, h - \frac{\sin. 2y \cos. ^2y}{2} h^2 -$$

$$\frac{\cos. 3y \cos. ^3y}{3} h^3 + \frac{\sin. 4y \cos. ^4y}{4} h^4 + \frac{\cos. 5y \cos. ^5y}{5} h^5 - \&c.$$

4. To develop log. $(x + h)$ according to the powers of h.

$$\log.(x + h) = \log .x + \text{M} \left(\frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \frac{h^4}{4x^4}\right) + \&c.$$

5. To develop tan. $(x + h)$ according to the powers of h.

$$\tan. (x + h) = \tan. x + \sec. ^2x \,.\, h + 2 \sec. ^2x \tan. x \frac{h^2}{1 \cdot 2} +$$

$$2 \sec.^2 x\, (1 + 3 \tan.^2x) \frac{h^3}{1 . 2 . 3} + \&c.$$

(33.) By means of the theorem of Taylor may be obtained a very commodious and useful form for the representation and subsequent determination of the differential coefficient, when the function is complicated, thus:

Let $u = \text{F}y$, y being any function of x, which we may represent by $y = fx$, and let it be required to find the expression for $\frac{du}{dx}$. Let x take the increment h, then since $y = fx$, the corresponding increment of y will, by Taylor's theorem, be

$$\frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} + \frac{d^3y}{dx^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c.$$

Call this increment k, then the corresponding increment of u will be

$$\text{F}(y + k) - \text{F}y = \frac{du}{dy} k + \frac{d^2u}{dy^2} \cdot \frac{k^2}{1 \cdot 2} + \&c.$$

that is, restoring the value of k

$$\text{F}(y + k) - \text{F}y = \frac{du}{dy} \left\{\frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} + \&c.\right\}$$

$$+ \frac{d^2u}{dy^2} \left\{\frac{dy}{dx} h + \frac{d^2y}{dx^2} \frac{h^2}{1 \cdot 2} + \&c.\right\}^2$$

$$+ \&c.$$

Hence dividing by the increment h of the independent variable, and taking the limit as usual, we have

$$\frac{dFy}{dx} = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} \therefore du = \frac{du}{dy} \cdot \frac{dy}{dx} dx.$$

It appears, therefore, that the differential coefficient $\frac{du}{dx}$ is found by differentiating the function, on the hypothesis that y is the independent variable, and then multiplying the coefficient thus obtained, by that derived from y considered as a function of x.

(34.) The following examples will suffice to illustrate this mode of finding the differential coefficient.

1. Let $u = a^y$ where $y = b^x$.

$$\text{1st. } \frac{du}{dy} = a^y \log. a, \text{ 2nd. } \frac{dy}{dx} = b^x \log. b,$$

$$\therefore \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = a^{b^x} b^x \log. a \log. b.$$

2. Let $u = \log. y$, where $y = \log. x$,

$$\therefore \frac{du}{dy} = \frac{1}{y}, \frac{dy}{dx} = \frac{1}{x},$$

$$\therefore \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{1}{x \log. x}$$

3. Let $u = \sin. (\frac{x}{a+x})^2$.

$$\text{Put } (\frac{x}{a+x})^2 = y \therefore \frac{du}{dy} = \frac{d \sin. y}{dy} = \cos. y, \frac{dy}{dx} = \frac{2ax}{(a+x)^3},$$

$$\therefore \frac{du}{dx} = \cos. (\frac{x}{a+x})^2 \cdot \frac{2ax}{(a+x)^3}.$$

4. Let $u = \cot. a^y$, y being $= \log. \frac{x}{\sqrt{a^2+x^2}}$,

$$\therefore \frac{du}{dy} = -\operatorname{cosec.}^2 a^y \cdot a^y \log. a, \frac{dy}{dx} = \frac{a^2}{x(a^2+x^2)}$$

$$\therefore \frac{du}{dx} = -\operatorname{cosec.}^2 a^y \cdot a^y \log. a \cdot \frac{a^2}{x(a^2+x^2)}.$$

(35.) Let us now take the more general function $u = F(p, q)$, p and q being functions of x, and suppose that when x becomes $x + h$, p and q become $p + k$ and $q + k'$. Call this latter q', then in consequence of the proposed change in the independent variable, the function will become $F(q', p + k)$, in which it is to be observed q'

enters as if it were a constant, since it is unaffected by the increment k of the variable p.

Hence by Taylor's theorem

$$F(q', p+k) = F(q', p) + \frac{d\,F(q', p)}{dp} k + \frac{d^2\,F(q', p)}{dp^2} \frac{k^2}{1 \cdot 2} + \&c. \ldots (1).$$

But if in $F(q', p)$ we substitute for q' its equal $q + k'$ we have

$$F(q', p) = F(p, q + k') = u + \frac{du}{dq} k' + \frac{d^2u}{dq^2} \frac{k'^2}{1 \cdot 2} + \&c. \ldots (2)$$

$$\therefore \frac{d\,F(q', p)}{dp} = \frac{du}{dp} + \frac{d \cdot \frac{du}{dq}}{dp} k' + \&c. \ldots (3),$$

and thus by continuing the differentiation may all the coefficients in (1) be developed according to the powers of k', but this first will be sufficient for our purpose.

Substitute in the first two terms of (1) their developments (2), (3) and we have

$$F(q + k', p + k) = u + \frac{du}{dq} k' + \&c. + \frac{du}{dp} k + \&c. \ldots (4)$$

But k being the increment of the function p, arising from x taking the increment h, and k' being the increment which the function q takes from the same cause, it follows that

$$k = \frac{dp}{dx} h + \frac{d^2p}{dx^2} \frac{h^2}{1 \cdot 2} + \&c., \quad k' = \frac{dq}{dx} h + \frac{d^2q}{dx^2} \frac{h^2}{1 \cdot 2} + \&c.$$

Hence by substitution in (4) we have finally

$$F(q + k', p + k) = u + \left\{\frac{du}{dq} \cdot \frac{dq}{dx} + \frac{du}{dp} \cdot \frac{dp}{dx}\right\} h + \&c.$$

$$\therefore \frac{du}{dx} = \frac{du}{dq} \cdot \frac{dq}{dx} + \frac{du}{dp} \cdot \frac{dp}{dx}.$$

(36.) Again: let there be three functions of x, viz. $v = F(p, q, r)$ then when x becomes $x + h$ let p, q, r become $p + k$, $q + k'$, $r + k''$ respectively, and put r' for the latter, then in the function $F(p + k, q + k', r')$, r' enters as a constant, hence as above

$$F(p + k, q + k', r') = u + \left\{\frac{du}{dq} \cdot \frac{dq}{dx} + \frac{du}{dp} \cdot \frac{dp}{dx}\right\} h + \&c.$$

where $u = F(p, q, r') \therefore$ putting $r + k''$ for r'.

$$u = v + \frac{dv}{dr} k'' + \&c., \frac{du}{dq} = \frac{dv}{dq} + \frac{d \cdot \frac{dv}{dr}}{dq} k'' + \&c.$$

$$\frac{du}{dp} = \frac{dv}{dp} + \frac{d \cdot \frac{dv}{dr}}{dp} k'' + \&c.$$

But $k'' = \frac{dr}{dx} h + \&c.$ consequently

$$\mathrm{F}\,(p+k, q+k', r+k'') = v + \left\{\frac{dv}{dr} \cdot \frac{dr}{dx} + \frac{dv}{dq} \cdot \frac{dq}{dx} + \frac{dv}{dp} \cdot \frac{dp}{dx}\right\} h + \&c.$$

$$\therefore \frac{dv}{dx} = \frac{dv}{dr} \cdot \frac{dr}{dx} + \frac{dv}{dq} \cdot \frac{dq}{dx} + \frac{dv}{dp} \cdot \frac{dp}{dx},$$

and so on for any number of functions. Hence the rule is to *differentiate the expression with regard to each of its constituent functions severally, as, if all the others were constants, their sum will be the required differential.*

Cor. If p is simply x then in the function $u = \mathrm{F}(x, q)$,

$$\frac{du}{dx} = \frac{du}{dx} + \frac{du}{dq} \cdot \frac{dq}{dx},$$

and in the function $u = \mathrm{F}\,(x, q, r)$,

$$\frac{du}{dx} = \frac{du}{dx} + \frac{du}{dq} \cdot \frac{dq}{dx} + \frac{du}{dr} \cdot \frac{dr}{dx}.$$

(37.) We must not confound here the $\frac{du}{dx}$ on the left, with that on the right, in these equations, for the former denotes the *total* differential coefficient, of which the latter forms but a part, and is therefore called a *partial* differential coefficient. It is to be regretted, however, that analysts are not agreed as to the best means of distinguishing total from partial differential coefficients, and accordingly in most works on the calculus the same symbol is applied indiscriminately to both; a circumstance likely to prove a frequent source of perplexity to the learner; and to avoid which we shall, throughout this volume, always distinguish the total differential coefficient by enclosing it in braces, so that the two equations above will be written thus:

$$\left\{\frac{du}{dx}\right\} = \frac{du}{dx} + \frac{du}{dq} \cdot \frac{dq}{dx}$$

$$\left\{\frac{du}{dx}\right\} = \frac{du}{dx} + \frac{du}{dq} \cdot \frac{dq}{dx} + \frac{du}{dr} \cdot \frac{dr}{dx}.$$

(38.) We shall now add a few examples, showing the application of the rule deduced in last article.

EXAMPLES.

1. Let $u = \cot. x^y \therefore \left\{\frac{du}{dx}\right\} = \frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx}$. Now

$$\frac{du}{dx} = -\operatorname{cosec.}^2 x^y \frac{dx^y}{dx} = -\operatorname{cosec.}^2 x^y\, yx^{y-1}$$

$$\frac{du}{dy} \cdot \frac{dy}{dx} = -\operatorname{cosec.}^2 x^y \frac{dx^y}{dx} = -\operatorname{cosec.}^2 x^y \,.\, x^y \log. x \frac{dy}{dx}$$

$$\therefore \left\{\frac{du}{dx}\right\} = -x^y \operatorname{cosec.}^2 x^y \left(\frac{y}{x} + \log. x \frac{dy}{dx}\right).$$

2. Let $u = \dfrac{a(x + \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x})}{\sqrt{x^2 + (x + \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x})^2}}$.

Put $x + \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x} = q \therefore u = F(x, q)$, and

$$\frac{du}{dx} = -\frac{aqx}{(x^2 + q^2)^{\frac{3}{2}}},$$

$$\frac{du}{dq} \cdot \frac{dq}{dx} = \frac{ax^2}{(x^2 + q^2)^{\frac{3}{2}}} \cdot \frac{dq}{dx}, \text{ and } \frac{dq}{dx} = 1 + \frac{1}{2\sqrt{x}} + \frac{1}{3x^{\frac{3}{2}}} + \frac{1}{4x^{\frac{3}{4}}};$$

hence

$$\left\{\frac{du}{dx}\right\} = \frac{ax^2 - aqx}{(x^2 + q^2)^{\frac{3}{2}}} \left(1 + \frac{1}{2x^{\frac{1}{2}}} + \frac{1}{3x^{\frac{2}{3}}} + \frac{1}{4x^{\frac{3}{4}}}\right).$$

3. Let $u = \log. \tan. \frac{x}{y}$, y being a function of x.

$$\frac{du}{dx} dx = \frac{d \tan. \frac{x}{y}}{\tan. \frac{x}{y}} = \frac{\sec.^2 \frac{x}{y} \cdot \frac{dx}{y}}{\tan. \frac{x}{y}} = \frac{dx}{y \sin. \frac{x}{y} \cos. \frac{x}{y}},$$

$$\frac{du}{dy} dy = \frac{d \tan. \frac{x}{y}}{\tan. \frac{x}{y}} = -\frac{\sec.^3 \frac{x}{y} \cdot \frac{xdy}{y^2}}{\tan. \frac{x}{y}} = -\frac{xdy}{y^2 \sin. \frac{x}{y} \cos. \frac{x}{y}}$$

hence

$$\left\{\frac{du}{dx}\right\} = \frac{y - x\dfrac{dy}{dx}}{y^2 \sin.\dfrac{x}{y}\ \cos.\dfrac{x}{y}}.$$

4. Let $u = \log.\,(x - a + \sqrt{x^2 - 2ax})$

$$\therefore \left\{\frac{du}{dx}\right\} = \frac{1}{\sqrt{x^2 - 2ax}}.$$

5. Let $u = (\cos.\, x)^{\sin.\, x}$

$$\therefore \left\{\frac{du}{dx}\right\} = (\cos.\, x)^{\sin.\, x}\,(\cos.\, x \log.\, \cos.\, x - \frac{\sin.^2 x}{\cos.\, x}).$$

Implicit Functions.

(39.) Hitherto we have considered *explicit* functions only, or those whose forms are supposed to be given. We shall now consider *implicit* functions, or those in which the relation between the independent variable x, and function y, is *implied* in an equation between the two, and which may be generally expressed by

$$u = \mathrm{F}\,(x, y) = 0.$$

The deductions in article (36) will enable us very readily to find the coefficient $\frac{dy}{dx}$ from such equations, without being under the necessity of solving them, a thing indeed often impossible.

If we turn to the corollary in the article just referred to, and substitute y for q, we find

$$\left\{\frac{du}{dx}\right\} = \frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx},$$

But here $u = \mathrm{F}\,(x, y) = 0$, therefore $\left\{\frac{du}{dx}\right\} = 0$, for $u' - u$ being always 0, $\frac{u' - u}{h}$ is always 0; hence,

$$\frac{du}{dx} + \frac{du}{dy} \cdot \frac{dy}{dx} = 0,$$

from which equation the differential coefficient is immediately determinable: it is

$$\frac{dy}{dx} = -\frac{du}{dx} \div \frac{du}{dy};$$

hence, *having transposed the terms all to one side of the equation, we must differentiate the expression as if y were a constant, and then divide the resulting coëfficient, taken with a contrary sign, by that derived from the same expression, on the supposition that x is a constant.*

EXAMPLES.

1. Let $u = y^2 - 2mxy + x^2 - a = 0$.

$$\therefore -\frac{du}{dx} = 2my - 2x, \frac{du}{dy} = 2y - 2mx \therefore \frac{dy}{dx} = \frac{my - x}{y - mx}.$$

2. Let $u = x^3 + 3axy + y^3 = 0$.

$$\therefore -\frac{du}{dx} = -3x^2 - 3ay, \frac{du}{dy} = 3ax + 3y^2 \therefore \frac{dy}{dx} = -\frac{x^2 + ay}{ax + y^2}.$$

If the second differential coefficient be required, we have

$$\frac{d^2y}{dx^2} = -\frac{(ax + y^2)(2x + a\frac{dy}{dx}) + (x^2 + ay)(a + 2y\frac{dy}{dx})}{(ax + y^2)^2}$$

or substituting for $\frac{dy}{dx}$ its value just found

$$\frac{d^2y}{dx^2} = -\frac{(ax + y^2)(2ax^2 + 2xy^2 - ax^2 - a^2y) + (x^2 + ay)(a^2x + ay^2 - 2x^2y - 2ay^2)}{(ax + y^2)^3}$$

$$= -\frac{2xy^4 + 6ax^2y^2 + 2x^4y - 2a^3xy}{(ax + y^2)^3}$$

$$= -\frac{2xy(y^3 + 3axy + x^3) - 2a^3xy}{(ax + y^2)^3},$$

that is, since $x^3 + 3axy + y^3 = 0$,

$$\frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax + y^2)^3}.$$

3. Let $my^3 - xy = m$ to develop y, according to the ascending powers of x,

$$-\frac{du}{dx} = y, \frac{du}{dy} = 3my^2 - x \therefore \frac{dy}{dx} = \frac{y}{3my^2 - x}$$

therefore, calling the successive differential coefficients p, q, r, &c.

$$q = \frac{y - 3my^2p - xp}{(3my^2 - x)^2},$$

$$r = -\frac{3my^2q + 2\cdot 3\,myp^2 + xq}{(3my^3 - x)^3} - \frac{(y - 3my^2p - xp)(12myp - 2)}{(3my^2 - x)^3}$$

&c. &c.

$$\therefore [y] = 1, [p] = \frac{1}{3m}, [q] = 0, [r] = -\frac{2}{3^3m^2}, \text{\&c.}$$

Therefore, by Maclaurin's theorem

$$y = 1 + \frac{x}{3m} - \frac{x^3}{3^4m^3} + \text{\&c.}$$

4. Let $y^3x - m^3(y + x) = 0$, to develop y according to the ascending powers of x.

Representing, as in last example, the successive differential coefficients by p, q, r, &c. we have

$$p = -\frac{y^3 - m^3}{3xy^2 - m^3} \therefore [p] = -1$$

$$q = -\frac{3y^2p}{3xy^2 - m^3} + (y^3 - m^3)\,.\,\frac{3y^2 + 2\cdot 3\,xyp}{(3xy^2 - m^3)^2} \therefore [q]=0,$$

$$\therefore [r] = -\frac{2\cdot 3\,yp^2}{3xy^2 - m^3} - \frac{2\cdot 3\,m^3(2yp + xp^2)}{(3xy^2 - m^3)^2} = 0,$$

$$\therefore [s] = -\frac{2\cdot 3\,p^3}{3xy^2 - m^3} - \frac{2\cdot 3\,.\,4\,m^3p^2}{(3xy^2 - m^3)^2} = -\frac{2\cdot 3\cdot 4}{m^3}$$

&c. &c.

Hence, by Maclaurin's theorem,

$$y = -x - \frac{x^4}{m^3} - \frac{3x^7}{m^6} - \text{\&c.}$$

5. Let $y^2 + 2xy + x^2 = a^2$, to find $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = -1.$$

6. Let $\frac{y^2}{(x - a)^2} = 2y\frac{\sqrt{x - b}}{x - a} - x + b$ to find $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{3x - 2b - a}{2\sqrt{x - b}}.$$

7. Let $y^3 - 3y + x = 0$, to develop y according to the ascending powers of x,

$$y = \frac{x}{3} + \frac{x^3}{3^4} + \frac{x^5}{3^6} + \text{\&c.}$$

8. Let $my^3x - y = m$, to develop y according to the ascending powers of x.

$$y = -m - m^4x - 3m^7x^2 - \&c.$$

9. Let $my^3 - x^3y = mx^3$, to develop y in a series of *descending** powers of x.

$$y = -m - \frac{m^4}{x^3} - \frac{3m^7}{x^6} - \&c.$$

CHAPTER V.

ON VANISHING FRACTIONS.

(40.) It is here proposed to determine the value of a fraction $\frac{\mathrm{F}x}{fx}$ in the case in which, by giving a particular value a to the variable, both numerator and denominator vanish, the fraction then becoming

$$\frac{\mathrm{F}a}{fa} = \frac{0}{0}.$$

As such a form can arise only from the circumstance of the same factor $x - a$ being common to both numerator and denominator, it is plain that if we can by any means eliminate this factor before our substitution of a for x, we shall then obtain the true value of the fraction.

Sometimes the vanishing factor is manifest at sight, and may be immediately expunged, as, for instance, in the fractions

$$\frac{(x-a)\,x}{bx - ab},\ \frac{x^2 - a^2}{(a - x)^2},\ \frac{a^2 - 2ax + x^2}{a^2 - x^2},\ \&c.$$

each of which becomes $\frac{0}{0}$ when $x = a$, and obviously contains the factor $x - a$ in both numerator and denominator. In these cases,

* This will be effected by substituting $\frac{1}{z}$ for x^3, which will transform the equation into $my^3z - y = m$, then developing y according to the ascending powers of z, and afterwards restoring the value of x.

therefore, we at once see that the values of the fractions when $x = a$ are, severally,

$$\frac{a}{b}, -\infty, 0, \&c.$$

In certain other cases the value, although not so easily seen as in the foregoing instances, may, nevertheless, be soon ascertained, by performing a few obvious transformations on the proposed fractions. Take the following example:

$$\frac{\sqrt{x} - \sqrt{a} + \sqrt{(x-a)}}{\sqrt{x^2 - a^2}}$$

which becomes $\frac{0}{0}$ when $x = a$.

This fraction is the same as

$$\frac{\sqrt{x} - \sqrt{a}}{\sqrt{x^2 - a^2}} + \frac{1}{\sqrt{x + a}}$$

and the first of these terms is the same as

$$\frac{x - a}{\sqrt{(x^2 - a^2)(\sqrt{x} + \sqrt{a})^2}} = \frac{\sqrt{x - a}}{\sqrt{(x + a)}\,(\sqrt{x} + \sqrt{a})}$$

and this when $x = a$ is $= 0$, therefore the value of the proposed fraction when $x = a$ is $\frac{1}{\sqrt{2a}}$.

(41.) But the most direct and general method of proceeding depends upon the differential calculus, and upon the development of functions, and the principal object of the present chapter is to explain this.

We shall premise the following lemma, viz. In the general development

$$F(x + h) = Fx + \frac{dFx}{dx} h + \frac{d^2Fx}{dx^2}\frac{h^2}{1 \cdot 2} + \&c.$$

it is impossible that any particular value given to x can cause Fx, and at the same time *all* the differential coefficients, to vanish.

For if such could be the case, then, for that particular value a, we should have

$$F(a + h) = 0,$$

whatever be the value of h; but $Fa = 0$, and therefore the prece-

ding equation can exist only when $h = 0$, whereas the hypothesis supposes it to exist independently of the value of h.

Let now, in the proposed fraction, x be changed into $x + h$, then, by developing both numerator and denominator, it becomes

$$\frac{Fx + F'xh + F''xh^2 + F'''xh^3 + \&c.}{fx + f'xh + f''xh^2 + f'''xh^3 + \&c.} \quad \ldots . \quad (1),$$

where $F'x$, $F''x$, &c, $f'x$, $f''x$. &c. are put for the successive differential coefficients divided by as many of the factors $1 \cdot 2 \cdot 3$, &c. as there are accents.

If in this we substitute a for x, then, since both Fx and fx vanish, the fraction becomes, after dividing numerator and denominator by h,

$$\frac{F'a + F''ah + F'''ah^2 + \&c.}{f'a + f''ah + f'''ah^2 + \&c.} \quad \ldots . \quad (2),$$

and this fraction when $h = 0$ must obviously be equal to $\frac{Fa}{fa}$, that is

$$\frac{Fa}{fa} = \frac{F'a}{f'a}.$$

If, however, both $F'a$ and $f'a$ are also 0, then, expunging these terms from the fraction (2) and dividing numerator and denominator again by h, we have, when $h = 0$,

$$\frac{Fa}{fa} = \frac{F''a}{f''a}$$

and so on, till we at length obtain for $\frac{Fa}{fa}$ a fraction of which the numerator and denominator do not both vanish, and such a fraction we eventually shall obtain by virtue of the preceding lemma.

Hence the following rule to determine the value of a fraction whose numerator and denominator both vanish when $x = a$, viz. *For the numerator and denominator substitute their first differential coefficients, their second differential coefficients, and so on till we obtain a fraction in which numerator and denominator do not both vanish, for $x = a$, this will be the true value of the vanishing fraction.*

EXAMPLES.

1. Required the value of $\frac{a^x - b^x}{x}$ when $x = 0$.

$$\frac{\text{F}'x}{f'x} = \log.\ a\ .\ a^x - \log.\ b\ .\ b^x \therefore \frac{\text{F}'a}{f'a} = \log.\ a - \log.\ b = \log.\ \frac{a}{b}$$

2. Required the value of $\dfrac{x^3 - 3x + 2}{x^4 - 6x^2 + 8x - 3}$ when $x = 1$.

$$\frac{\text{F}'x}{f'x} = \frac{3x^2 - 3}{4x^3 - 12x + 8} \therefore \frac{\text{F}'a}{f'a} = \tfrac{0}{0} \therefore$$

differentiating again

$$\frac{\text{F}''}{f''}\ \frac{\text{F}'x}{f'x} = \frac{6x}{12x^2 - 12} \therefore \frac{\text{F}''a}{f''a} = \tfrac{1}{0} = \infty\,.$$

3. Required the value of $\dfrac{1 - \sin.\ x + \cos.\ x}{\sin.\ x + \cos.\ x - 1}$ when $x = 90°$.

$$\frac{\text{F}'x}{f'x} = -\frac{\cos.\ x + \sin.\ x}{\cos.\ x - \sin.\ x} \therefore \frac{\text{F}'a}{f'a} = 1.$$

4. Required the value of

$$\frac{x + x^2 - (n+1)^2\, x^{n+1} + (2n^2 + 2n - 1)\, x^{n+2} - n^2 x^{n+3}}{(1-x)^3}$$

when $x = 1$.

$$\frac{\text{F}'x}{f'x} =$$

$$\frac{1 + 2x - (n+1)^3 x^n + (n+2)\,(2n^2 + 2n - 1) x^{n+1} - n^2\,(n+3) x^{n+2}}{3\,(1-x)^2}$$

$$\therefore \frac{\text{F}'a}{f'a} = \frac{0}{0}$$

$$\frac{\text{F}''x}{f''x} =$$

$$\frac{2 - n(n+1)^3 x^{n-1} + (n+1)(2n^3 + 6n^2 + 3n - 2) x^n - (n+2)(n^3 + 3n^2) x^{n+1}}{6\,(1-x)}$$

$$\therefore \frac{\text{F}''a}{f''a} = \frac{0}{0}$$

$$\frac{\text{F}'''x}{f'''x} =$$

$$\frac{-(n^2 - n)(n+1)^3 x^{n-2} + (n^2+n)(2n^3 + 6n^2 + 3n - 2) x^{n-1} (n+1)(n+2)(n^3 + 3n^2) x^n}{-6}$$

$$\therefore \frac{\text{F}'''a}{f'''a} = \frac{n\,(n+1)\,(2n+1)}{6} = \frac{\text{F}a}{fa}.$$

5. Required the value of $\frac{a - \sqrt{a^2 - x^2}}{x^2}$ when $x = 0$.

$$\frac{F''a}{f''a} = \frac{1}{2a}.$$

6. Required the value of $\frac{x \sin. x - 90^\circ}{\cos. x}$ when $x = 90^\circ$.

$$\frac{F'a}{f'a} = -1.$$

7. Required the value of $\frac{1 - x}{\cot. x \frac{\pi}{2}}$ when $x = 1$.

$$\frac{F'a}{f'a} = \frac{2}{\pi}.$$

8. Required the value of $\frac{a^n - x^n}{\log. a - \log. x}$ when $x = a$.

$$\frac{F'a}{f'a} = na^n.$$

(42.) If, in the application of the foregoing rule, we happen to arrive at a differential coefficient, which becomes infinite for the proposed value $x = a$, we must conclude that the development according to Taylor's Theorem is impossible for that particular value of the variable; and that, therefore, the rule which is founded on the possibility of this development becomes inapplicable. The process, however, to be adopted in such cases is still analogous to that above, depending upon the development of the numerator and denominator of the proposed fraction; but here this development must be sought for by the common algebraical methods.

As before, let $\frac{Fx}{fx}$ be a fraction which becomes $\frac{0}{0}$ when we change x into a. Substitute $a + h$ for x, and let the terms of the fraction be developed according to the increasing powers of h, either by involution, the extraction of roots, or some other algebraical process, then we shall have

$$\frac{F(a+h)}{f(a+h)} = \frac{Ah^{\alpha} + Bh^{\beta} + \&c.}{A'h^{\alpha'} + B'h^{\beta'} + \&c.}$$

α and α' being the smallest exponents in each series, β and β' the next in magnitude, and so on. Now these three cases present themselves, viz.

$$1^\circ.\ \alpha > \alpha';\ 2^\circ.\ \alpha = \alpha';\ 3^\circ.\ \alpha < \alpha'.$$

In the first case by dividing the two terms of the fraction by $h^{\alpha'}$, and then supposing $h = 0$, there results

$$\frac{\mathrm{F}a}{fa} = \frac{0}{\mathrm{A}'} = 0.$$

In the second case the result of the same process is

$$\frac{\mathrm{F}a}{fa} = \frac{\mathrm{A}}{\mathrm{A}'}.$$

In the third case, by dividing the two terms of the fraction by h^{α}, and then supposing $h = 0$, the result is

$$\frac{\mathrm{F}a}{fa} = \frac{\mathrm{A}}{0} = \infty.$$

It appears from these results that the development of the numerator and denominator need not be carried beyond the first term, or that involving the lowest exponent of h,* and according as the exponent in the numerator is greater than, equal to, or less than that in the denominator, will the true value of the fraction be 0, finite, or infinite. We have, therefore, the following rule:

Substitute $a + h$ for x, in the proposed fraction. Find the term containing the lowest exponent of h, in the development of the numerator, and that containing the lowest exponent of h in the development of the denominator. If the former exponent be greater than this latter, the true value of the fraction will be 0, if less, it will be infinite. But if these exponents are equal, divide the coefficient of the term in the numerator by the coefficient of that in the denominator, and the true result will be obtained.

This method, which is applicable in all cases, may frequently be employed advantageously, even where the preceding rule applies.

EXAMPLES.

9. Required the value of $\dfrac{(x^2 - 3ax + 2a^2)^{\frac{2}{3}}}{(x^3 - a^3)^{\frac{1}{2}}}$ when $x = a$.

* The first term which actually *appears* in the development is of course meant here. Those which may vanish in consequence of the coefficient vanishing not being considered.

Substituting $a + h$ for x, we have

$$\frac{\mathrm{F}\,(a+h)}{f\,(a+h)} = \frac{h^{\frac{2}{3}}\,(h-a)^{\frac{2}{3}*}}{h^{\frac{1}{2}}\,(3a^2 + 3ah + h^2)^{\frac{1}{2}}} = \frac{(-ah)^{\frac{2}{3}} + \&c.}{(3a^2h)^{\frac{1}{2}} + \&c.}$$

Since the exponent of h in the numerator exceeds that in the denominator, we have

$$\frac{\mathrm{F}a}{fa} = 0.$$

10. Required the value of $\dfrac{\sqrt{x} - \sqrt{a} + \sqrt{x-a}}{\sqrt{x^2 - a^2}}$ when $x = a$ (see p. 53.)

Substituting $a + h$ for x.

$$\frac{\mathrm{F}\,(a+h)}{f\,(a+h)} = \frac{(a+h)^{\frac{1}{2}} - a^{\frac{1}{2}} + h^{\frac{1}{2}}}{h^{\frac{1}{2}}\,(2a+h)^{\frac{1}{2}}} = \frac{h^{\frac{1}{2}} + \&c.}{(2ah)^{\frac{1}{2}} + \&c.}$$

$$\therefore \frac{\mathrm{F}a}{fa} = \frac{1}{(2a)^{\frac{1}{2}}}.$$

11. Required the value of $\dfrac{(x^2 - a^2)^{\frac{3}{2}} + x - a}{(1 + x - a)^3 - 1}$ when $x = a$.

Substituting $a + h$ for x.

$$\frac{\mathrm{F}\,(a+h)}{f\,(a+h)} = \frac{h^{\frac{2}{3}}\,(2a+h)^{\frac{3}{2}} + h}{(1+h)^3 - 1} = \frac{h + \&c.}{3h + \&c.}$$

$$\therefore \frac{\mathrm{F}a}{fa} = \tfrac{1}{3}.$$

12. Required the value of $\dfrac{a\,(4a^3 + 4x^3)^{\frac{1}{3}} - ax - a^2}{(2a^2 + 2x^2)^{\frac{1}{2}} - a - x}$ when $x = a$.

Substituting $a + h$ for x.

$$\frac{\mathrm{F}\,(a+h)}{f\,(a+h)} = \frac{a\,(8a^3 + 12a^2h + 12ah^2 + 4h^3)^{\frac{1}{3}} - 2a^2 - ah}{(4a^2 + 4ah + 2h^2)^{\frac{1}{2}} - 2a - h}$$

* To develop this according to the ascending powers of h we must write it thus: $(-a + h)^{\frac{2}{3}}$ and apply the binomial theorem when we have the series $(-a)^{\frac{2}{3}} + \frac{2}{3}\,a^{-\frac{1}{3}}\,h + \&c.$

which, by actually extracting the roots indicated,

$$= \frac{a\,(2a + h + \frac{h^2}{2a} + \&c. - 2a - h)}{2a + h + \frac{h^2}{4a} + \&c. - 2a - h}$$

$$= \frac{a\,(\frac{h^2}{2a} + \&c.)}{\frac{h^2}{4a} + \&c.} \therefore \frac{\mathrm{F}a}{fa} = \frac{\frac{1}{2}}{\frac{1}{4a}} = 2a.$$

This example is perhaps more easily performed by differentiation, according to the first rule: thus

$$\frac{\mathrm{F}'x}{f'x} = \frac{a\,(4a^3 + 4x^3)^{-\frac{2}{3}}\,4x^2 - a}{(2a^2 + 2x^2)^{-\frac{1}{2}}\,2x - 1} \therefore \frac{\mathrm{F}'a}{f'a} = \frac{0}{0}$$

$$\frac{\mathrm{F}''x}{f''x} = \frac{-a\,(4a^3 + 4x^3)^{-\frac{5}{3}}\,32x^4 + a\,(4a^3 + 4x^3)^{-\frac{2}{3}}\,8x}{-(2a^2 + 2x^2)^{-\frac{3}{2}}\,4x^2 + 2\,(2a^2 + 2x^2)^{-\frac{1}{2}}},$$

$$\therefore \frac{\mathrm{F}''a}{f''a} = \frac{2}{\frac{1}{a}} = 2a.$$

(43.) Having thus seen how to determine the value of any fraction of which the numerator and denominator become each 0 for particular values of the variable, we readily perceive how the value may be found when particular substitutions make the numerator and denominator each infinite. For if $\frac{\mathrm{F}a}{fa} = \frac{\infty}{\infty}$ then obviously

$$\frac{\mathrm{F}a}{fa} = \frac{\frac{1}{fa}}{\frac{1}{\mathrm{F}a}} = \frac{0}{0}$$

So that if we find, by the preceding methods, the value of this last fraction, the value of the proposed fraction will be also obtained.

The following example will illustrate this.

13. Required the value of $\dfrac{\tan.\,(\frac{1}{2}\pi\,.\,\frac{x}{a})}{x^2\,a^{-1}\,(x^2 - a^2)^{-1}}$ when $x = a$.

In this case the fraction takes the form $\frac{\infty}{\infty}$, therefore,

$$\frac{\frac{1}{fx}}{\frac{1}{Fx}} = \frac{a\,(x^2 - a^2)\,x^{-2}}{\text{cot.}\,(\frac{1}{2}\,\pi\,.\,\frac{x}{a})} \therefore \frac{F'x}{f'x} = \frac{2a^3\,x^{-3}}{-\,\text{cosec.}^2\,(\frac{1}{2}\,\pi\,.\,\frac{x}{a})\,\frac{\pi}{2a}}$$

$$\therefore \frac{F'a}{f'a} = \frac{2}{-\frac{\pi}{2a}} = -\frac{4a}{\pi}$$

(44.) By the same principles we may also find the true value of a *product* consisting of two factors, which for a particular value of the variable becomes the one 0 and the other ∞. For if $Fa = 0$ and $fa = \infty$, then,

$$Fa \times fa = \frac{Fa}{\frac{1}{fa}} = \frac{0}{0}$$

We shall give an example of this.

14. Required the value of the product $(1 - x)\,\text{tan.}\,(\frac{1}{2}\,\pi x)$ when $x = 1$.

In this case the first factor becomes 0 and the second ∞.

$$\frac{Fx}{\frac{1}{fx}} = \frac{1 - x}{\frac{1}{\text{tan.}\,(\frac{1}{2}\,\pi x)}} = \frac{1 - x}{\text{cot.}\,(\frac{1}{2}\,\pi\,x)} \therefore F'x \times f'x = \frac{1}{\text{cosec.}^2\,(\frac{1}{2}\,\pi\,x)\frac{\pi}{2}},$$

$$\therefore F'a \times f'a = \frac{1}{\frac{1}{2}\pi} = \frac{2}{\pi}.$$

(45.) And finally, by the same principles, the true value of the difference of two functions may be ascertained in the case where the substitution of a particular value for the variable causes each of them to become infinite.

For if $Fa = \infty$ and $fa = \infty$, then

$$Fa - fa = \frac{\frac{1}{fa} - \frac{1}{Fa}}{\frac{1}{Fa \times fa}} = \frac{0}{0}$$

The following example belongs to this case:

15. Required the true value of the difference x tan. $x - \frac{1}{2}\pi$ sec. x, when $x = 90°$.

$$\frac{\frac{1}{fx} - \frac{1}{Fx}}{\frac{1}{Fx \times fx}} = \frac{\frac{1}{\frac{1}{2}\pi \text{ sec. } x} - \frac{1}{x \text{ tan. } x}}{\frac{1}{x \text{ tan. } x \times \frac{1}{2}\pi \text{ sec. } x}} = \frac{x \text{sin.} x - \frac{1}{2}\pi}{\text{cos. } x}*$$

by substituting $\frac{1}{\text{cos. } x}$ for sec. x, and then dividing numerator and denominator by $\frac{1}{\frac{1}{2}\pi\, x \text{ tan. } x}$

$$\therefore F'x - f'x = \frac{x \text{ cos. } x + \text{sin. } x}{- \text{sin. } x} \therefore Fa - fa = -1.$$

It should be remarked that in this, as well indeed as in the preceding cases, the transformation requisite to reduce the expression to the form $\frac{0}{0}$ in many instances at once presents itself to the mind, when of course it will be necessary to recur to the preceding formulas. The example just given is one of these instances, for since tan. $x = \frac{\text{sin. } x}{\text{cos. } x}$, and sec. $x = \frac{1}{\text{cos. } x}$, the proposed expression at once reduces to

$$\frac{x \text{ sin. } x - \frac{1}{2}\pi}{\text{cos. } x}$$

which is the required form.

(46.) We shall terminate this chapter with a few miscellaneous examples for the exercise of the student.

16. Required the value of $\frac{x^n - 1}{x - 1}$, when $x = 1$.

Ans. n.

17. Required the value of $\frac{ax^2 + ac^2 - 2acx}{bx^2 - 2bcx + bc^2}$, when $x = c$.

Ans. $\frac{a}{b}$.

18. Required the value of $\frac{x^3 - ax^2 - a^2x + a^3}{x^2 - a^2}$, when $x = a$.

Ans. 0.

* This is obtained by multiplying the last fraction above and below by x tan. $x \times \frac{1}{2}\pi$ sec. x, then writing $\frac{\text{sin. } x}{\text{cos. } x}$ for tan. x, and $\frac{1}{\text{cos. } x}$ for sec. x. *Ed.*

19. Required the value of $\dfrac{(2a^3x - x^4)^{\frac{1}{3}} - a\,(a^2x)^{\frac{1}{3}}}{a - (ax^3)^{\frac{1}{4}}}$, when $x = a$. *Ans.* $\dfrac{16a}{9}$.

20. Required the value of $\dfrac{1 - x + \log. x}{1 - (2x - x)^{2\frac{1}{2}}}$, when $x = 1$.
Ans. -1.

21. Required the value of $\dfrac{x^x - x}{1 - x + \log. x}$, when $x = 1$.
Ans. -2.

22. Required the value of $\dfrac{\tan. x - \sin. x}{\sin. x^3}$, when $x = 0$.
Ans. $\frac{1}{2}$.

23. Required the value of $\dfrac{x \log. x - (x - 1)}{(x - 1) \log. x}$, when $x = 1$.
Ans. $\frac{1}{2}$.

24. Required the value of $\dfrac{(x^2 - a^2)^{\frac{3}{2}}}{(x - a)^{\frac{3}{2}}}$, when $x = a$.
Ans. $2a^{\frac{3}{2}}$.

25. Required the value of $\dfrac{x}{x - 1} - \dfrac{1}{\log. x}$, when $x = 1$.
Ans. $\frac{1}{2}$.

26. Required the value of $\dfrac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4}$, when $x = a$.
Ans. ∞.

27. Required the value of $\dfrac{1}{\log. x} - \dfrac{x}{\log. x}$, when $x = 1$.
Ans. -1.

28. Required the value of $\dfrac{x^2 - a^2}{x^2} \,.\, \tan. \dfrac{\pi x}{2a}$, when $x = a$.
Ans. $-\dfrac{4}{\pi}$.

29. Required the value of $\dfrac{a\,(1 - x)}{\cot. \frac{1}{2} \pi\, x}$, when $x = 1$.
Ans. $\dfrac{2a}{\pi}$.

30. Required the value of $\dfrac{e^x - e^{\sin. x}}{x - \sin. x}$, when $x = 0$.
Ans. 1.

CHAPTER VI.

ON THE MAXIMA AND MINIMA VALUES OF FUNCTIONS OF A SINGLE VARIABLE.

*(47.) In any function $y = Fx$ let the independent variable take a particular value $x = a$, as also a preceding and succeeding value $x = a - h$ and $x = a + h$, then the corresponding values of the function, arranged according to those of the variable, will be

$$F(a - h), Fa, F(a + h);$$

and if a be such that for any finite value of h, however small, and for all intermediate values between this and 0, the middle value Fa *exceeds* that on each side, the value $x = a$ is said to render the function a *maximum;* but if the middle value continue *less* than that on each side between the same limits of h, the value $x = a$ is said to render the function a *minimum;*† so that we are not always to understand by the expression, *maximum* value of a function, the *greatest* value such function can possibly take the term being of more comprehensive meaning, applying to *every* state of the function which exceeds its immediately preceding and succeeding state. In like manner, the *minimum* value of a function does not always imply the *least* possible value of such function, but equally characterizes every state of the function which is less than its immediately preceding and succeeding state.

Before proceeding to the general method of determining the values of x, necessary to render any proposed function a maximum or a minimum, we must premise this lemma:

If the function $F(a + h)$ *be developed according to the ascending powers of h, a value so small may be given to h that any proposed term in the series shall exceed the sum of all that follow.*

* The maximum value of any function is that, which is greater than those which *immediately* precede and follow it; and the minimum value is that, which is less than those which *immediately* precede and follow it. It is proper to observe that the same function may have several maxima and minima values. *Ed.*

† This definition of a maximum and a minimum is but a slight alteration of that given by Dr. Lardner in his *Differential Calculus*, p. 103.

(48.) Let Ah^{α} be any proposed term in the development, and let Bh^{β}, Ch^{γ}, &c. be those which follow, each exponent being greater than that which precedes it. We are to prove that h may be taken so small that

$$Ah^{\alpha} > h^{\beta}\,(B + Ch^{\gamma-\beta} + Dh^{\delta-\beta} + \&c.)$$

Putting S for the sum of the series within the parentheses, it is obvious that h may be taken so small that $Sh^{\beta-\alpha}$ may be less than any proposed quantity A, and that therefore if h' be such a value we must have

$$Ah'^{\alpha} > Sh'^{\beta}$$

which establishes the proposition. As $Sh^{\beta-\alpha}$ is less than A for $h = h'$, the expression continues less than A for every value of h less than h'.

(49.) Let us now inquire by what means we may determine those values of x which render any proposed function Fx a maximum or a minimum. In order to do this, let x be changed into $x \pm h$, then by Taylor's theorem

$$F\,(x \pm h) = Fx \pm \frac{dy}{dx}\,h + \frac{d^2y}{dx^2}\,\frac{h^2}{1 \cdot 2} \pm \frac{d^3y}{dx^3}\,\frac{h^3}{1 \cdot 2 \cdot 3} + \frac{d^4y}{dx^4}\,\frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} \pm \&c.$$

Now if $x = a$ render the proposed function a *maximum*, then there exists for h some finite value h', such that for all the intermediate values between this and 0 we have

$$Fa > F\,(a \pm h),$$

and, consequently,

$$\pm \left[\frac{dy}{dx}\right] h + \left[\frac{d^2y}{dx^2}\right]\frac{h^2}{1 \cdot 2} \pm \left[\frac{d^3y}{dx^3}\right]\frac{h^3}{1 \cdot 2 \cdot 3} + \&c. < 0 \quad . \; . \; . \quad (1).$$

But if this value render the function a *minimum*, then, for all the intermediate values of h between $h = h'$ and $h = 0$, we have

$$Fa < F\,(a \pm h)$$

and, consequently,

$$\pm \left[\frac{dy}{dx}\right] h + \left[\frac{d^2y}{dx^2}\right]\frac{h^2}{1 \cdot 2} \pm \left[\frac{d^3y}{dx^3}\right]\frac{h^3}{1 \cdot 2 \cdot 3} + \&c. > 0 \quad . \; . \; . \quad (2).$$

It has, however, been proved above, that a value may be given to h small enough to render the first term in each of the series (1) and (2) greater than the sum of all the other terms, and that this first term will continue greater for all other values of h between this small value and 0, so that, for each of these values of h, the sign belonging to the sum of the whole series is the same as that of the first term; it is impossible, therefore, that either of the conditions (1) or (2) can exist for *both* $+ [\frac{dy}{dx}]\, h$ and $- [\frac{dy}{dx}]\, h$, unless $[\frac{dy}{dx}] = 0$; we conclude, therefore, that those values of x only can render the function a maximum or minimum which fulfil the condition

$$\frac{dy}{dx} = 0;$$

expunging, therefore, the first term from each of the series, (1), (2), we have, in the case of a maximum, the condition

$$[\frac{d^2y}{dx^2}]\frac{h^2}{1 \cdot 2} \pm [\frac{d^3y}{dx^3}]\frac{h^3}{1 \cdot 2 \cdot 3} + \&c \angle 0 \; . \; . \; . \; (3),^*$$

and in the case of a minimum,

$$[\frac{d^2y}{dx^2}]\frac{h^2}{1 \cdot 2} + [\frac{d^3y}{dx^3}]\frac{h^3}{1 \cdot 2 \cdot 3} + \&c. \mathrel{7} 0 \; . \; . \; . \; (4).$$

Now the former of these conditions cannot exist for any of the values of h between $h = h'$ and $h = 0$, by virtue of the foregoing principle, unless $[\frac{d^2y}{dx^2}]$ is negative, nor can the latter condition exist unless $[\frac{d^2y}{dx^2}]$ is positive, that is, supposing that these coefficients do not vanish from the series (3) and (4).

We may infer, therefore, that of the values of x which satisfy the condition $\frac{dy}{dx} = 0$, those among them that also satisfy the condition $\frac{d^2y}{dx^2} \angle 0$ belong to maximum values of the function, while those fulfilling the condition $\frac{d^2y}{dx^2} \mathrel{7} 0$ belong to minimum values of the function. It is possible, however, that some of the values derived from the equation $\frac{dy}{dx} = 0$ may, when substituted for x in $\frac{d^2y}{dx^2}$, cause this

* See Note (C′).

coefficient to vanish, in which case the conditions (1), (2), become

$$\pm \left[\frac{d^3y}{dx^3}\right] \frac{h^3}{1 \cdot 2 \cdot 3} + \left[\frac{d^4y}{dx^4}\right] \frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} \pm \&c. \angle 0 \quad . \quad . \quad . \quad (5),$$

and

$$\pm \left[\frac{d^3y}{dx^3}\right] \frac{h^3}{1 \cdot 2 \cdot 3} + \left[\frac{d^4y}{dx^4}\right] \frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} \pm \&c. \,7\, 0 \quad . \quad . \quad . \quad (6),$$

which are both impossible unless $\left[\frac{d^3y}{dx^3}\right] = 0$, for reasons similar to those assigned above, and, unless, also $\left[\frac{d^4y}{dx^4}\right] \angle 0$ in the case of a maximum, and $\left[\frac{d^4y}{dx^4}\right] \,7\, 0$ in the case of a minimum; that is, on the supposition that this coefficient does not vanish from the series (5) and (6). If, however, this coefficient does vanish, then, for reasons similar to those assigned in the preceding cases, the following coefficient $\frac{d^5y}{dx^5}$ must also vanish, and the condition of maximum will then be $\left[\frac{d^6y}{dx^6}\right] \angle 0$, and the condition of minimum $\left[\frac{d^6y}{dx^6}\right] \,7\, 0$, and so on.

It hence appears, that to determine what values of x correspond to the maxima and minima values of the function $y = \mathrm{F}x$, we must proceed as follows:

Determine the real roots of the equation $\frac{dy}{dx} = 0$, and substitute them one by one in the following coefficients $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. stopping at the first, which does not vanish. If this is of an *odd* order, the root that we have employed is not one of those values of x that renders the function either a maximum or a minimum; but if it is of an *even* order, then, according as it is negative or positive, will the root employed correspond to a maximum or to a minimum value of the function.

(50.) It must however be remarked, that, should any of the roots of the equation $\frac{dy}{dx} = 0$ cause the first of the following coefficients, which does not vanish, to become infinite, we cannot apply to such roots the foregoing tests for distinguishing the maxima from the

minima, because the true development of the function for any such value of x begins to differ in form from Taylor's development, at that term which is thus rendered infinite (4), so that we cannot infer, from Taylor's series, whether the power of h, which ought to enter this is odd or even.

In a case of this kind, therefore, we must find, by actual involution, extraction, &c. the true term that ought to supply the place of that rendered infinite in Taylor's series for $x = a$. If this term take an odd power of h, or, rather, if its sign change with the sign of h, then $x = a$ does not render the function either a maximum or a minimum; but if the sign does not change with that of h, then the value of x renders the function a maximum or a minimum, according as the sign of this term is negative or positive.

To illustrate this case, suppose the function were

$$y = b + (x - a)^{\frac{5}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{5}{3}(x - a)^{\frac{2}{3}}$$

$$\frac{d^2y}{dx^2} = \frac{10}{9}(x - a)^{-\frac{1}{3}}$$

Now the equation $\frac{dy}{dx} = 0$ gives $x = a$, so that if any value of x could render the proposed function a maximum or a minimum, this most likely would be it. By substituting this value of x in $\frac{d^2y}{dx^2}$ the result is infinite, and we cannot infer the state of the function from this coefficient; therefore, substituting $a \pm h$ for x in the proposed, we have

$$\text{F}(a \pm h) = b \pm h^{\frac{5}{3}}$$

and, as $h^{\frac{5}{3}}$ obviously changes its sign when h does, we conclude that the function proposed admits of neither a maximum nor a minimum value.

Again, let

$$y = b + (x - a)^{\frac{4}{3}}$$

$$\therefore \frac{dy}{dx} = \frac{4}{3}(x - a)^{\frac{1}{3}}$$

$$\frac{d^2y}{dx^2} = \frac{4}{9}(x - a)^{-\frac{2}{3}}$$

The equation $\frac{dy}{dx} = 0$ gives $x = a$, a value which causes $\frac{d^2y}{dx^2}$ to become infinite; therefore, substituting $a \pm h$ for x in the proposed, we have

$$F(a \pm h) = b = h^{\frac{4}{3}}$$

and, as the sign of $h^{\frac{4}{3}}$ is positive whatever be the sign of h, we conclude that the value $x = a$ renders the function a minimum.

(51.) There remains to be considered one more case to which the general rule is not applicable, and which, like the preceding, arises from the failure of Taylor's theorem. We have hitherto examined only those values of x for which Taylor's development is possible, as far at least as the first power of h, but we cannot say that among those values of x, which would render the coefficient of this first power *infinite*, there may not be some which cause the function to fulfil the conditions of maxima or minima; therefore, before we can conclude in any case that the values of x, deduced from the condition $\frac{dy}{dx} = 0$, comprise among them *all* those which can render the function a maximum or minimum, we must examine those values of x arising from the condition $\frac{dy}{dx} = \infty$ by substituting each of these $\pm h$ for x in the proposed equation, and observing which of the results agree with the conditions of maxima and minima in (47).

(52.) If the function that y is of x be *implicitly* given, that is, if

$$u = F(x, y) = 0;$$

then, by (39), we have, for the differential coefficient,

$$\frac{dy}{dx} = -\frac{du}{dx} \div \frac{du}{dy} \quad . \; . \; . \; (1),$$

and therefore, when $\frac{dy}{dx} = 0$, we must have $\frac{du}{dx} = 0$; hence, the values corresponding to maxima and minima, are determinable from the two equations*

* Other values may be implied in the condition $\frac{dy}{dx} = \infty$, which leads to $\frac{du}{dy} = 0$, but to ascertain which of these are applicable would require us to solve the equation for y.

$$\left.\begin{aligned} u &= 0 \\ \frac{du}{dx} &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . (2).$$

Having found from these values of x that *may* render y a maximum or a minimum,* as also the corresponding values of y itself, we must substitute them for x and y in $\frac{d^2y}{dx^2}$, when those values of y will be maxima that render this coefficient negative, and those will be minima that render it positive. But those values that cause it to vanish, belong neither to maxima nor to minima, unless the same values cause also $\frac{d^3y}{dx^3}$ to vanish, and so on.

The second differential coefficient may be readily derived from (1), for, putting for brevity

$$\frac{dy}{dx} = -\frac{\text{M}}{\text{N}},$$

we have

$$\frac{d^2y}{dx^2} = -\frac{\text{N}\left(\frac{d\text{M}}{dx} + \frac{d\text{M}}{dy} \cdot \frac{dy}{dx}\right) - \text{M}\left(\frac{d\text{N}}{dx} + \frac{d\text{N}}{dy} \cdot \frac{dy}{dx}\right)}{\text{N}^2}$$

which, because $\frac{\text{M}}{\text{N}} = 0$, becomes for the particular values of x resulting from this condition,

$$\left[\frac{d^2y}{dx^2}\right] = -\left[\frac{d^2u}{dx^2}\right] \div \left[\frac{du}{dy}\right] \quad . \quad . \quad . \quad (3).$$

By differentiating the above expression for $\frac{d^2y}{dx^2}$ we shall find

$$\left[\frac{d^3y}{dx^3}\right] = -\left[\frac{d^3u}{dx^3}\right] \div \left[\frac{du}{dy}\right] \quad . \quad . \quad . \quad (4),$$

and so on.

(53.) Before we proceed to apply the foregoing theory to examples, we shall state a few particulars that may, in many instances, be serviceable in abridging the process of finding maxima and minima.

* Garnier, at p. 271 of his *Calcul Différentiel*, says, that, by means of the equations (2) "on obtient les valeurs de x et y par lesquelles F (x, y) devient ou peut devenir maximum ou minimum;" but this is evidently a mistake, since, by hypothesis, F (x, y) is always $= 0$.

1. if the proposed function appears with a constant factor, such factor may be omitted. Thus, calling the function Ay, the first differential coefficient will be A$\frac{dy}{dx}$, and A$\frac{dy}{dx} = 0$ leads to $\frac{dy}{dx} = 0$, also $\frac{1}{A\frac{dy}{dx}} = 0$ leads to $\frac{1}{\frac{dy}{dx}} = 0$, so that A may be expunged from the function.

2. Whatever value of x renders a function a maximum or minimum, the same value must obviously render its square, cube, and every other power, a maximum or minimum; so that when a proposed function is under a radical, this may be removed. The rational function may, however, become a maximum or a minimum for more values of x than the original root; indeed, all values of x which render the rational function *negative* will render every even root of it imaginary; such values, therefore, do not belong to that root; moreover, if the rational function be $= 0$, when a maximum, the corresponding value of the variable will be inadmissible in any even root, because the contiguous values of the function must be negative.

3. The value $x = \infty$ can never belong to a maximum or minimum, inasmuch as it does not admit of both a preceding and *succeeding* value.

EXAMPLES.

(54.) 1. To determine for what values of x the function

$$y = a^4 + b^3x - c^2 x^2$$

becomes a maximum or minimum,

$$\frac{dy}{dx} = b^3 - 2c^2x, \frac{d^2y}{dx^2} = -2c^2.$$

From the second equation it appears that, whatever be the values of x, given by the condition $\frac{dy}{dx} = 0$, they must all belong to maxima.

From $b^3 - 2c^2x = 0$, we get $x = \frac{b^3}{2c^2}$; hence

$$\text{when } x = \frac{b^3}{2c^2} \therefore y = a^4 + \frac{b^6}{4c^2}, \text{ a maximum.}$$

The equation $\frac{dy}{dx} = \infty$ would give, in the present case, $x = \infty$, a value which is inadmissible (53).

2. To determine the maxima and minima values of the function

$$y = 3a^2x^3 - b^4x + c^5$$

$$\frac{dy}{dx} = 9a^2x^2 - b^4, \frac{d^2y}{dx^2} = 18a^2x$$

putting

$$9a^2x^2 - b^4 = 0 \therefore x = \pm \frac{b^2}{3a}$$

Substituting each of these values in $\frac{d^2y}{dx^2}$ we infer from the results that

$$\text{when } x = \frac{b^2}{3a} \ldots y = c^5 - \frac{2b^6}{9a}, \text{ a min.}$$

$$x = -\frac{b^2}{3a} \ldots y = c^5 + \frac{2b^6}{9a}, \text{ a max.}$$

3. To determine the maxima and minima values of the function

$$y = \sqrt{2ax}.$$

Omitting the radical

$$u = 2ax \therefore \frac{du}{dx} = 2a,$$

as this can never become 0 or ∞, we infer that the function has no maximum or minimum value.

4. To determine the maximum and minimum values of the function

$$y = \sqrt{4a^2x^2 - 2ax^3}.$$

Omitting the radical and the constant factor $2a$ (53),

$$u = 2ax^2 - x^3,$$

$$\therefore \frac{du}{dx} = 4ax - 3x^2, \frac{d^2u}{dx^2} = 4a - 6x,$$

$$\therefore x(4a - 3x) = 0 \therefore x = 0, \text{ or } x = \frac{4a}{3}.$$

Substituting each of these values in $\frac{d^2u}{dx^2}$, the results are $4a$ and $-4a$; hence

$$\text{when } x = 0 \ldots y = 0, \text{ a minimum,}$$

$$x = \frac{4a}{3} \quad . \; . \quad y = \frac{8}{3}a^2, \text{ maximum.}$$

If, instead of the preceding, the example had been

$$y = \sqrt{2ax^3 - 4a^2x^2},$$

we should have had

$$\frac{du}{dx} = 3x^2 - 4ax, \quad \frac{d^2u}{dx^2} = 6x - 4a.$$

$$\therefore x\,(3x - 4a) = 0, \therefore x = 0, \text{ or } x = \frac{4a}{3}$$

the same values as before; but the first corresponds here to a *maximum*, since it makes $\frac{d^2u}{dx^2}$ negative; this value, therefore, must, by (53), be rejected. If, indeed, we substitute $0 \pm h$ for x, in the proposed function, it becomes

$$y = \sqrt{-4a^2h^2 \mp 2ah^3},$$

where h may be taken so small as to cause the expression under the radical to be negative for all values of h between this and 0.

5. To determine the maxima and minima values of the function

$$y = a + \sqrt[3]{a^3 - 2a^2x + ax^2}.$$

If y is a maximum or minimum, $y - a$ will be so; therefore, transposing the a, and omitting the radical (53),

$$u = a^3 - 2a^2x + ax^2$$

$$\frac{du}{dx} = -2a^2 + 2ax, \quad \frac{d^2u}{dx^2} = 2a,$$

$$\therefore -2a^2 + 2ax = 0 \;\therefore x = a,$$

$$\therefore \text{when } x = a \; . \; . \; . \; y = a, \text{ a minimum.}$$

6. To determine the maxima and minima values of the function

$$y = \frac{a^2x}{(a - x)^2}.$$

In solving this example we shall employ a principle that is often found useful, when the proposed function is a fraction with a denominator more complex than the numerator. Instead of the function itself we shall take its reciprocal, which will give us a more simple form, and it is plain that the maxima and minima values of the reciprocal of a

function correspond respectively to the minima and maxima of the function itself. Omitting, then, the constant a^3, and, taking the reciprocal, we have

$$u = \frac{a^2 - 2ax + x^2}{x} = \frac{a^2}{x} - 2a + x$$

$$\therefore \frac{du}{dx} = -\frac{a^2}{x^2} + 1, \frac{d^2u}{dx^2} = \frac{2a^2}{x^3}$$

$$\therefore -\frac{a^2}{x^2} + 1 = 0 \therefore x = \pm a \therefore \left[\frac{d^2u}{dx^2}\right] = \pm \frac{2}{a},$$

hence $x = a$ makes u a minimum, and $x = -a$ makes it a maximum, therefore

$$\text{when } x = a \ldots y = \infty, \text{ a maximum,}$$

$$x = -a \ldots y = -\tfrac{1}{2} a, \text{ a minimum.}$$

7. To determine the maxima and minima values of the function

$$y = b + \sqrt[3]{(x-a)^5}.$$

Omitting b and the radical

$$u = (x-a)^5$$

$$\therefore \frac{du}{dx} = 5(x-a)^4, \frac{d^2u}{dx^2} = 4 \cdot 5(x-a)^3$$

$$\therefore 5(x-a)^4 = 0 \therefore x = a \therefore \left[\frac{d^2u}{dx^2}\right] = 0.$$

As this coefficient vanishes, we must proceed to the following, which however all contain $x - a$, and therefore vanish, till we come to $\frac{d^5u}{dx^5} = 2 \cdot 3 \cdot 4 \cdot 5$; as therefore the first coefficient which does not vanish is of an odd order, the function does not admit of a maximum or a minimum.

8. To determine the maxima and minima values of the function

$$y = x^x$$

$$\frac{dy}{dx} = x^x(1 + \log. x), \frac{d^2y}{dx^2} = x^x \{\frac{1}{x} + (1 + \log. x)^2\}.$$

The factor x^x can never become 0, therefore

$$(1 + \log. x) = 0 \therefore \log. x = -1.$$

$$\therefore x = c^{-1} = \frac{1}{e}$$

$$\therefore \left[\frac{d^2y}{dx^2}\right] = \left(\frac{1}{e}\right)^{\frac{1}{e}} . e$$

$$\therefore \text{when } x = \frac{1}{e},\ x^x = \left(\frac{1}{e}\right)^{\frac{1}{e}},\ \text{a minimum.}$$

9. To determine the maxima and minima values of y in the function

$$u = x^3 - 3axy + y^3 = 0$$

$$\frac{du}{dx} = 3x^2 - 3ay \therefore (52)$$

$$\left.\begin{array}{r} x^3 - 3axy + y^3 = 0 \\ 3x^2 - 3ay = 0 \end{array}\right\} \therefore y = \frac{x^2}{a} \therefore x^6 - 2a^3x^3 = 0$$

$$\therefore x = 0 \text{ or } x = a\sqrt[3]{2} \therefore (52)$$

$$\left[\frac{d^2y}{dx^2}\right] = -\left[\frac{d^2u}{dx^2}\right] \div \left[\frac{du}{dy}\right] = -[6x] \div \left[3\frac{x^4}{a^2} - 3ax\right] = -\left[\frac{2a^2}{x^3 - a^3}\right]$$

$$= \frac{2}{a} \text{ or } -\frac{2}{\sqrt[3]{2} - 1}$$

$$\therefore \text{when } x = 0 \ldots\ldots y = 0, \text{ a minimum.}$$

$$x = a\sqrt[3]{2} \ldots\ldots y = a\sqrt[3]{4}, \text{ a maximum.}$$

10. To divide a given number a, into two parts, such that the product of the mth power of the one and the nth power of the other shall be the greatest possible.

Let x be one part, then $a - x$ is the other, and

$$y = x^m (a - x)^n = \text{maximum},$$

$$\therefore \frac{dy}{dx} = mx^{m-1}(a - x)^n - nx^m (a - x)^{n-1}$$

$$= x^{m-1}(a - x)^{n-1}\{ma - (m + n)x\} = 0,$$

$$\therefore x = 0, \text{ or } a - x = 0, \text{ or}$$

$$ma - (m + n)\,x = 0,$$

which give

$$x = 0, x = a, x = \frac{ma}{m + n}.$$

The first and second of these values are inadmissible, because the number is not *divided* when $x = 0$ or when $x = a$.

Substituting the third value in

$$\frac{d^2y}{dx^2} = x^{m-2}\,(a - x)^{n-2}\,\{(ma - (m + n)\,x)^2 - m\,(a - x)^2 - nx^2\}$$

we have

$$\left[\frac{d^2y}{dx^2}\right] = - [x]^{m-2}\,[a - x]^{n-2}\,\{m\,[a - x]^2 + nx^2\}$$

which is negative because each factor is positive, hence the two required parts are

$$\frac{ma}{m + n} \text{ and } \frac{na}{m + n} \text{ being to each other as } m \text{ to } n.$$

Cor. If $m = n$ the parts must be equal.

An easier solution to this problem may be obtained as follows:

Put $\frac{m}{n} = p$ and determine x so that we may have

$$u = x^p\,(a - x) = \text{A maximum},$$

$$\therefore \frac{du}{dx} = px^{p-1}\,(a - x) - x^p$$

$$= x^{p-1}\,\{pa - (p + 1)\,x\} = 0,$$

$$\therefore x = 0 = \text{or } pa - (p + 1)\,x = 0 \;\therefore\; x = \frac{pa}{p + 1}.$$

This last value substituted in

$$\frac{d^2u}{dx^2} = x^{p-2}\,\{pa - (p + 1)\,x\} - (p + 1)\,x^{p-2}$$

causes the first term to vanish; the result is therefore negative, so that $x = \frac{pa}{p + 1} = \frac{ma}{m + n}$ corresponds to a maximum value of u, and therefore (53) to a maximum value of $u^n = x^m\,(a - x)^n$.

Another easy mode of solution is had by using logarithms, for it is

obvious that since the logarithm of any number increases with the number, when this number is the greatest possible, its logarithm will be so also.

$$\therefore m \log. x + n \log. (a - x) = max.$$

$$\therefore \frac{du}{dx} = \frac{m}{x} - \frac{n}{a - x} = 0 \therefore ma - (m + n)\, x = 0$$

$$\therefore x = \frac{ma}{m + n}$$

as before.

The expression for the second differential coefficient is $-(m + n)$ showing that the foregoing value of x renders the logarithmic expression a maximum.

11. To divide a number a, into so many equal parts, that their continued product may be the greatest possible.

It is obvious from the corollary to the last example, that the parts must be equal, for the product of any two unequal parts of a number is less than that of equal parts.

Let x be the number of factors, then,

$$(\frac{a}{x})^x = a, \text{ maximum},$$

$$\therefore \log. (\frac{a}{x})^x = x \log. (\frac{a}{x}) = \text{a maximum},$$

$$\therefore \log. \frac{a}{x} - 1 = 0$$

$$\therefore \frac{a}{x} = \log.^{-1} 1 = e \therefore x = \frac{a}{e}*$$

hence the proposed number must be divided by the number $e = 2{\cdot}718281828$.

12. To determine those conjugate diameters of an ellipse which include the greatest angle.

Call the principal semi-diameters of the ellipse a, b, the sought semi-conjugates x, x' and the sine of the angle they include y. Then (*Anal. Geom.*)

* There is obviously no necessity to recur to the second differential coefficient to ascertain whether this value render the function a maximum or a minimum, since it is plain that there is no minimum unless each of the parts may be 0.

$$x^2 + x'^2 = a^2 + b^2 \therefore x' = \sqrt{a^2 + b^2 - x^2}$$

$$xx'y = ab \therefore y = \frac{ab}{xx'}$$

$$\therefore y = \frac{ab}{x\sqrt{a^2 + b^2 - x^2}} = max.$$

Omitting the constant ab, inverting the function (ex. 6.) and squaring, we have

$$u = a^2x^2 + b^2x^2 - x^4 = max.$$

$$\therefore \frac{du}{dx} = 2a^2x + 2b^2x - 4x^3 = 0,$$

$$\therefore x = 0,\ a^2 + b^2 - 2x^2 = 0 \therefore x^2 = \frac{a^2 + b^2}{2} = \frac{x^2 + x'^2}{2}.$$

The first of these values is inadmissible, from the second we find that

$$x^2 = x'^2$$

hence the conjugates are equal. For the second differential coefficient we have

$$\frac{d^2u}{dx^2} = 2a^2 + 2b^2 - 12x^2$$

$$\therefore [\frac{d^2u}{dx^2}] = -4\ (a^2 + b^2).$$

This being negative, shows that $x = \sqrt{\frac{a^2 + b^2}{2}}$ corresponds to a maximum value of u, or to a minimum value of y, so that the conjugates here determined, include an angle whose *sine* is the least possible; and this happens when the angle itself is the greatest possible (being obtuse), as well as when it is the least possible.

13. To divide an angle θ into two parts, such that the product of the nth power of the sine of one part of the mth power of the sine of the other part may be the greatest possible.

Let x be one part, then $\theta - x$ is the other, and

$$\sin.^n x \,.\, \sin.^m (\theta - x) = \text{maximum},$$

$$\therefore n \log. \sin. x + m \log. \sin. (\theta - x) = \text{maximum},$$

$$\therefore \frac{n \cos. x}{\sin. x} - \frac{m \cos. (\theta - x)}{\sin. (\theta - x)} = 0,$$

$$\therefore n \tan.\ (\theta - x) = m \tan.\ x,$$

$$\therefore n : m :: \tan.\ x : \tan.\ (\theta - x),$$

$$\therefore n + m : n - m :: \tan.\ x + \tan.\ (\theta - x) : \tan.\ x - \tan.\ (\theta - x),$$

$$:: \sin.\ \theta : \sin.\ (2x - \theta),*$$

$$\therefore \sin.\ (2x - \theta) = \frac{n - m}{n + m} \sin.\ \theta,$$

which determines x.

14. Given the hypothenuse of a right-angled triangle to determine the other sides, when the surface is the greatest possible.

Call the hypothenuse a, and one of the sides x, then the other will be $\sqrt{a^2 - x^2}$ and the area of the triangle will be

$$\frac{x}{2} \sqrt{a^2 - x^2} = \text{maximum}.$$

$$\therefore u = a^2x^2 - x^4 = \text{maximum}.$$

$$\therefore \frac{du}{dx} = 2a^2x - 4x^3 = 0 \therefore x = 0 \text{ or } x = \frac{a}{\sqrt{2}}.$$

Substituting the second value in

$$\frac{d^2u}{dx^2} = 2a^2 - 12x^2$$

the result being negative, shows that the above value of x corresponds to a maximum. Therefore the required sides are each $\frac{a}{\sqrt{2}}$.

15. To determine the maxima and minima values of the function

$$y = x^3 - 18x^2 + 96x - 20.$$

when $x = 4 \ldots . y = 356$ a maximum.

$x = 8 \ldots . y = 128$ a minimum.

16. To determine a number x, such that the xth root may be the greatest possible.

Ans. $x = e = 2{\cdot}71828 \ldots .$

17. What fraction is that which exceeds its mth power by the greatest possible number?

Ans. $\sqrt[m-1]{\frac{1}{m}}$.

* Dr. Gregory's Trigonometry, p. 47, Equation (S).

18. Given the equation

$$y^2 - 2mxy + x^2 = a^2,$$

to determine the maxima and minima values of y.

$$\text{When } x = \frac{ma}{\sqrt{1 - m^2}} \ldots y = \frac{a}{\sqrt{1 - m^2}}, \text{ a maximum,}$$

$$x = \frac{-ma}{\sqrt{1 - m^2}} \ldots y = \frac{-a}{\sqrt{1 - m^2}}, \text{ a minimum.}$$

19. Given the position of a point between the sides of a given angle to draw through it a line so that the triangle formed may be the least possible.

Ans. The line must be bisected by the point.

20. The equation of a certain curve is $a^2y = ax^2 - x^3$ required its greatest and least ordinates.

$$\text{When } x = \tfrac{2}{3}a \ldots y = \text{maximum,}$$

$$x = 0 \ldots y = \text{minimum.}$$

21. To divide a given angle θ less than 90° into two parts, x and $\theta - x$, such that $\tan.^n x \,.\, \tan.^m (\theta - x)$ may be the greatest possible.

$$\tan. (2x - \theta) = \frac{n - m}{n + m} \tan. \theta.$$

22. To determine the greatest parabola that can be formed by cutting a given right cone.*

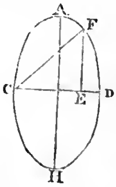
SCHOLIUM.

(55.) It will be proper, before terminating the present chapter, to apprize the student that in the application of the theory of maxima and minima to geometrical inquiries, care must be taken that we do not adopt results inconsistent with the geometrical restrictions of the problem. We know, indeed, from the first principles of Analytical Geometry, that when the geometrical conditions of a problem are translated into an algebraical formula, that formula is not necessarily restricted to those conditions, but, in addition to all the possible solutions of the problem, may also furnish others that belong merely to the analytical expression, and have no geometrical signification.† If,

* It will be shown hereafter that a parabola is equal to $\frac{2}{3}$ of a rectangle of the same base and altitude.

† See the Analytical Geometry.

therefore, among these latter solutions there be any belonging to maxima or minima, they are inadmissible in the application of this theory to Geometry. The following example is given by *Simpson*, at art 47 of his *Treatise on Fluxions*, to illustrate this.



From the extremity C of the minor axis of an ellipse to draw the longest line to the curve. Suppose F to be the point to which the line must be drawn, and call the abscissa CE, x then the geometrical restrictions of this variable are such that its values must always lie between the limits $x = 0$ and $x = 2b$, a and b denoting the semi-axes.

By the equation of the curve.

$$EF^2 = y^2 = \frac{a^2}{b^2}(2bx - x^2)$$

$$\therefore CF^2 = u = x^2 + \frac{a^2}{b^2}(2bx - x^2) = \text{maximum}.$$

$$\therefore \frac{du}{dx} = 2\left(x + \frac{a^2}{b} - \frac{a^2}{b^2}x\right) = 0$$

$$\therefore x = \frac{a^2b}{a^2 - b^2}$$

and since $\frac{d^2u}{dx^2} = 2\left(1 - \frac{a^2}{b^2}\right)$ it follows that the foregoing expression for x renders u a maximum for all values of b less than a, and a minimum for all values of b greater than a. Hence if the relation between a and b be such that $\frac{a^2b}{a^2 - b^2}$ may exceed $2b$, the analytical expression for CF will admit of a maximum value, although such value, not coming within the geometrical restrictions of the problem, is inadmissible. If the relation between a and b be such that $\frac{a^2b}{a^2 - b^2} = 2b$, that is, if $a^2 = 2b^2$, the solution will be valid, and in the ellipse whose axes are thus related CD will be the longest line that can be drawn from C, agreeably to the analytical determination, and the solution will always be valid if the axes of the ellipse are related so that $\frac{a^2b}{a^2 - b^2}$ is not greater than $2b$, which leads to the condition $2b^2$ not greater than a^2.

CHAPTER VII.

ON THE DIFFERENTIATION AND DEVELOPMENT OF FUNCTIONS OF TWO INDEPENDENT VARIABLES.

Differentiation of functions of two independent variables.

(56.) Let $z = F(x, y)$ be a function of two independent variables; then since in consequence of this independence, however either be supposed to vary, the other will remain unchanged: the function ought to furninsh two differential coefficients; the one arising from ascribing a variation to x and the other from ascribing a variation to y, y entering the first coefficient as if it were a constant, and x entering the second as if it were a constant. The differential coefficient arising from the variation of x is expressed thus, $\frac{dz}{dx}$; and that arising from the the variation of y thus, $\frac{dz}{dy}$; and these are called the *partial differential coefficients*, being analogous to those bearing the same name considered in chapter IV. We have seen, in functions of a single variable, that if that variable take an increment, and the function be developed, what we have called the differential coefficient will be the coefficient of the first power of the increment in that development; so here, as will be shortly shown, the partial differential coefficients are no other than the coefficients of the first power of the increments in the development of the function from which they are derived. As to the *partial differentials* they are obviously $\frac{dz}{dx}dx$ and $\frac{dz}{dy}dy$ and hence we call $\frac{dz}{dx}dx + \frac{dz}{dy}dy$ the *total differential* of the function, that is,

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy,$$

and we immediately see that this form becomes the same as that

given in chapter IV. for the differential of F (x, q) as soon as we suppose y to be a function of x, for we then have

$$\left\{\frac{dz}{dx}\right\} = \frac{dz}{dx} + \frac{dz}{dy} \cdot \frac{dy}{dx},$$

as indeed we ought.

In a similar manner, if the function consist of a greater number of independent variables as $u = \mathrm{F}(x, y, z, \&c.)$ we should necessarily have as many independent differentials, of which the aggregate would be the *total* differential of the function, that is

$$du = \frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{dz} dz + \&c.$$

Hence, whether the variables are dependent or independent, we infer, generally, that

The total differential of any function is the sum of the several partial differentials arising from differentiating the function relatively to each variable in succession, as if all the others were constants.

We shall add but few examples in functions of independent variables, seeing that the process is exactly the same as for functions of dependent variables.

$$d(x + y) = dx + dy$$

$$d \,.\, xy = ydx + xdy$$

$$d \frac{x}{y} = \frac{ydx - xdy}{y^2}$$

$$d \,.\, \frac{ay}{\sqrt{x^2 + y^2}} = \frac{ax^2dy - ayxdx}{(x^2 + y^2)^{\frac{3}{2}}}$$

$$d \tan.^{-1} \frac{x}{y} = \frac{ydx - xdy}{y^2 + x^2}$$

$$d \frac{y}{3y^2 - x} = \frac{ydx - 3y^2dy - xdy}{(3y^2 - x)^2}$$

$$d \,.\, a^x b^y c^z = a^x b^y c^z (dx \log. a + dy \log. b + dz \log. c)$$

$$d \log. \tan. \frac{x}{y} = \frac{ydx - xdy}{y^2 \sin. \frac{x}{y} \cos. \frac{x}{y}} = \frac{2(ydx - xdy)}{y^2 \sin. \frac{2x}{y}}$$

$$dy^x = y^x \log. ydx + y^{x-1} xdy.$$

(57.) If the function that x, y is of z is given implicitly, that is by the equation

$$u = \mathrm{F}\,(x, y, z) = 0,$$

then

$$\left\{\frac{du}{dx}\right\}^* = 0 \text{ and } \left\{\frac{du}{dy}\right\} = 0,$$

but (39),

$$\left\{\frac{du}{dx}\right\} = \frac{du}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx} = 0$$

$$\left\{\frac{du}{dy}\right\} = \frac{du}{dy} + \frac{du}{dz} \cdot \frac{dz}{dy} = 0$$

$$\therefore du = \left(\frac{du}{dx} + \frac{du}{dz} \cdot \frac{dz}{dx}\right) dx + \left(\frac{du}{dy} + \frac{du}{dz} \cdot \frac{dz}{dy}\right) dy = 0$$

$$= \left\{\frac{du}{dx}\right\} dx + \left\{\frac{du}{dy}\right\} dy.$$

Thus: let $\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 - 1 = 0$,

$$\therefore \left\{\frac{du}{dx}\right\} = 2\mathrm{A}x + 2\mathrm{C}z\frac{dz}{dx} = 0$$

$$\left\{\frac{du}{dy}\right\} = 2\mathrm{B}y + 2\mathrm{C}z\frac{dz}{dy} = 0$$

$$\therefore du = \left(\mathrm{A}x + \mathrm{C}z\frac{dz}{dx}\right) dx + \left(\mathrm{B}y + \mathrm{C}z\frac{dz}{dy}\right) dy = 0.$$

(58.) If $u = \mathrm{F}z$, z being a function of x and y, the two differential coefficients are (33)

$$\frac{du}{dx} = \frac{du}{dz} \cdot \frac{dz}{dx}, \frac{du}{dy} = \frac{du}{dz} \cdot \frac{dz}{dy}$$

and the total differential is, therefore,

* The brackets are employed here for the same purpose as at (37), viz. to imply the *total differential coefficient derived from u, considered as a function of a single variable.* This form it will be necessary to adopt whenever u contains, besides x, other variables that are functions of x, provided we wish to express the *total* coefficient with respect to x. No ambiguity can arise from our calling these same coefficients *partial* in one sense, and *total* in another. They are *partial* coefficients in relation to the *whole* variation of u, but they are *total* coefficients as far as that variable is concerned whose differential forms the denominator; and it may be remarked here, once for all, that when we enclose a differential coefficient in brackets, we mean the *total differential coefficient* to be understood, arising from considering the function, whose differential is the numerator, as simply a function of the variables whose differentials form the denominator.

$$du = \frac{du}{dz} \cdot \frac{dz}{dx} dx + \frac{du}{dz} \cdot \frac{dz}{dy} dy.$$

Now it is worthy of notice, that *the ratio of the two partial differential coefficients is independent of F*, so that this may be any function whatever. Thus

$$\frac{du}{dx} \div \frac{du}{dy} = \frac{du}{dz} \cdot \frac{dz}{dx} \div \frac{du}{dz} \cdot \frac{dz}{dy} = \frac{dz}{dx} \div \frac{dz}{dy}$$

which is an important property, since it enables us to eliminate any arbitrary function F of a determinate function $f(x, y)$ of two variables. We shall often have occasion to employ it in discussing the theory of curve surfaces. By means of this property too we may readily ascertain whether an expression containing two variables is a function of any proposed combination of those variables. For, calling this combination z and the function u, we shall merely have to ascertain whether or not the above condition exists, or, which is the same thing, whether or not the condition

$$\frac{du}{dx} \cdot \frac{dz}{dy} - \frac{du}{dy} \cdot \frac{dz}{dx} = 0$$

exists. For instance, suppose we wished to know whether $u = x^4 + 2x^2y^2 + y^4$ is a function of $z = x^2 + y^2$.

Here

$$\frac{du}{dx} = 4x^3 + 4xy^2, \frac{du}{dy} = 4x^2y + 4y^3; \frac{dz}{dy} = 2y, \frac{dz}{dx} = 2x;$$

$$\therefore \frac{du}{dx} \cdot \frac{dz}{dy} - \frac{du}{dy} \cdot \frac{dz}{dx} = (4x^3 + 4xy^2)\, 2y - (4x^2y + 4y^3)\, 2x = 0;$$

consequently, since the proposed condition exists, we infer that *u is a function of x.*

We shall now proceed to apply Taylor's theorem to functions of two independent variables.

Development of Functions of two Independent Variables.

(59.) In the function $z = F(x, y)$ suppose x takes the increment h, the function will become $F(x + h, y)$, y remaining unchanged, since it is independent of x, then, by Taylor's theorem,

$$F(x + h, y) = z + \frac{dz}{dx} h + \frac{d^2z}{dx^2} \cdot \frac{h^2}{1 \cdot 2} + \frac{d^3z}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} + \text{\&c.} \ldots (1).$$

But if y also take an increment k, then z will become

$$z + \frac{dz}{dy} k + \frac{d^2z}{dy^2} \cdot \frac{k^2}{1 \cdot 2} + \frac{d^3z}{dy^3} \cdot \frac{k^3}{1 \cdot 2 \cdot 3} + \&c.$$

so that in the expression (1) we must for $\frac{dz}{dx}$ substitute

$$\frac{dz}{dx} + \frac{d \cdot \frac{dz}{dy}}{dx} k + \frac{d \cdot \frac{d^2z}{dy^2}}{dx} \cdot \frac{k^2}{1 \cdot 2} + \frac{d \cdot \frac{d^3z}{dy^3}}{dx} \cdot \frac{k^3}{1 \cdot 2 \cdot 3} + \&c.$$

for $\frac{d^2z}{dx^2}$

$$\frac{d^2z}{dx^2} + \frac{d^2 \cdot \frac{dz}{dy}}{dx^2} k + \frac{d^2 \cdot \frac{d^2z}{dy^2}}{dx^2} \cdot \frac{k^2}{1 \cdot 2} + \frac{d^2 \cdot \frac{d^3z}{dy^3}}{dx^2} \cdot \frac{k^3}{1 \cdot 2 \cdot 3} + \&c.$$

for $\frac{d^3z}{dx^3}$

$$\frac{d^3z}{dx^3} + \frac{d^3 \cdot \frac{dz}{dy}}{dx^3} k + \frac{d^3 \cdot \frac{d^2z}{dy^2}}{dx^3} \cdot \frac{k^2}{1 \cdot 2} + \frac{d^3 \cdot \frac{d^3z}{dy^3}}{dx^3} \cdot \frac{k^3}{1 \cdot 2 \cdot 3} + \&c.$$

and so on. Before, however, we actually make these substitutions, we shall, for abridgment, write

$$\frac{d^2z}{dy\,dx} \text{ for } \frac{d \cdot \frac{dz}{dy}}{dx}, \quad \frac{d^3z}{dy^2\,dx} \text{ for } \frac{d \cdot \frac{d^2z}{dy^2}}{dx} \text{ and generally } \frac{d^{q+p}z}{dy^q dx^p}$$

$$\text{for } \frac{d^p \cdot \frac{d^qz}{dy^q}}{dx^p}$$

this last expression implying that after having determined the qth differential coefficient of the function z relatively to the variable y, the pth differential coefficient of this is taken relatively to the other variable x. Hence, the result of the proposed substitutions in (1) will be

$$F(x + h, y + k) =$$

$$\left.\begin{array}{l} z + \dfrac{dz}{dx}\,h \\ \ \dfrac{dz}{dy}\,k \\ \\ \\ \end{array}\right| \left.\begin{array}{l} + \dfrac{d^2z}{dx^2}\cdot\dfrac{h^2}{1\cdot 2} \\ \ \dfrac{d^2z}{dy\,dx}\,kh \\ \ \dfrac{d^2z}{dy^2}\cdot\dfrac{k^2}{1\cdot 2} \\ \\ \end{array}\right| \left.\begin{array}{l} + \dfrac{d^3z}{dx^3}\cdot\dfrac{h^3}{1\cdot 2\cdot 3} \\ \ \dfrac{d^3z}{dy\,dx^2}\cdot\dfrac{kh^2}{1\cdot 2} \\ \ \dfrac{d^3z}{dy^2\,dx}\cdot\dfrac{k^2h}{1\cdot 2} \\ \ \dfrac{d^3z}{dy^3}\cdot\dfrac{k^3}{1\cdot 2\cdot 3} \end{array}\right| + \&c.$$

The general term of the development being

$$\frac{d^{q+p}z}{dy^q\,dx^p}\cdot\frac{k^qh^p}{(1\cdot 2\ldots q)(1\cdot 2\ldots p)}.$$

If in the proposed function $z = \mathrm{F}\,(x, y)$ we had supposed y to vary first, then, instead of (1), whe should have had

$$\mathrm{F}\,(x, y + k) = z + \frac{dz}{dy}\,k + \frac{d^2z}{dy^2}\cdot\frac{k^2}{1\cdot 2} + \frac{d^3z}{dy^3}\cdot\frac{k^3}{1\cdot 2\cdot 3} + \&c.\ \ldots\ (2).$$

But, if x take the increment h, z will become

$$z + \frac{dz}{dx}\,h + \frac{d^2z}{dx^2}\cdot\frac{h^2}{1\cdot 2} + \frac{d^3z}{dx^3}\cdot\frac{h^3}{1\cdot 2\cdot 3} + \&c.$$

and, therefore, we must substitute, agreeably to the foregoing notation for $\dfrac{dz}{dy}$

$$\frac{dz}{dy} + \frac{d^2z}{dx\,dy}\,h + \frac{d^3z}{dx^2\,dy}\cdot\frac{h^2}{1\cdot 2} + \frac{d^4z}{dx^3\,dy}\cdot\frac{h^3}{1\cdot 2\cdot 3} + \&c.$$

for $\dfrac{d^2z}{dy^2}$

$$\frac{d^2z}{dy^2} + \frac{d^3z}{dx\,dy^2}\,h + \frac{d^4z}{dx^2\,dy^2}\cdot\frac{h^2}{1\cdot 2} + \frac{d^5z}{dx^3\,dy^2}\cdot\frac{h^3}{1\cdot 2\cdot 3} + \&c.$$

for $\dfrac{d^3z}{dy^3}$

$$\frac{d^3z}{dy^3} + \frac{d^4z}{dx\,dy^3}\,h + \frac{d^5z}{dx^2\,dy^3}\cdot\frac{h^2}{1\cdot 2} + \frac{d^6z}{dx^3\,dy^3}\cdot\frac{h^3}{1\cdot 2\cdot 3} + \&c.$$

and so on; so that the development would be

$$\mathrm{F}\,(x + h, y + k) =$$

$$\begin{array}{l|l|l|l}
z + \frac{dz}{dx}h & + \frac{d^2z}{dx^2} \cdot \frac{h^2}{1\,.\,2} & + \frac{d^3z}{dx^3} \cdot \frac{h^3}{1 \cdot 2 \cdot 3} & + \&c. \\
\quad \frac{dz}{dx}k & \quad \frac{d^2z}{dx\,dy} \,.\, hk & \quad \frac{d^3z}{dx^2\,dy} \cdot \frac{h^2k}{1 \cdot 2} & \\
 & \quad \frac{d^2z}{dy^2} \cdot \frac{k^2}{1\,.\,2} & \quad \frac{d^3z}{dx\,dy^2} \cdot \frac{hk^2}{1 \cdot 2} & \\
 & & \quad \frac{d^3z}{dy^3} \cdot \frac{k^3}{1 \cdot 2 \cdot 3} &
\end{array}$$

the general term being

$$\frac{d^{p+q}z}{dx^p\,dy^q} \cdot \frac{h^p\,.\,k^q}{(1 \cdot 2 \ldots p)\ (1 \cdot 2 \ldots q)}.$$

As this development must be identical with that exhibited above, we have, by equating the like powers of h and k,

$$\frac{d^2z}{dy\,dx} = \frac{d^2z}{dx\,dy}, \ \frac{d^3z}{dy\,dx^2} = \frac{d^3z}{dx^2\,dy},$$

and generally

$$\frac{d^{q+p}z}{dy^q\,dx^p} = \frac{d^{p+q}z}{dx^p\,dy^q},$$

we conclude, therefore, thatif we first determine the qth differential coefficient relatively to the variable y, and then the pth differential coefficient of this relatively to the variable x, the final result will be the same as if we first determine the pth differential coefficient relatively to x, and then the qth differential coefficient of this relatively to y; so that *the result is the same in whichever order the differentiations are performed.*

(60.) We see from the foregoing development, that the partial differential coefficients of the first order are the coefficients of h and k, the first power of the increments, so that the term containing these first powers is in this respect analogous to that containing the first power of the increment in the development of functions of a single variable, and, by a very slight transformation, it will be seen that the same analogy extends throughout a l the terms of the two developments. For the development just given may be put under the form

$$F\,(x + h, y + k) =$$

$$z$$

$$+ \left(\frac{dz}{dx}h + \frac{dz}{dy}k\right)$$

$$+ \frac{1}{2}\left(\frac{d^2z}{dx^2}h^2 + 2\frac{d^2z}{dx\,dy}hk + \frac{d^2z}{dy^2}k^2\right)$$

$$+ \frac{1}{2\cdot 3}\left(\frac{d^3z}{dx^3}h^3 + 3\frac{d^3z}{dx^2\,dy}h^2k + 3\frac{d^3z}{dx\,dy^2}h\,k^2 + \frac{d^3z}{dx^3}k^3\right)$$

$$+ \quad \&c.$$

where the partial differential coefficients in each term are identical with those which appear in the differential of the preceding term, as the actual differentiation shows, thus:

$$dz = \frac{dz}{dx}dx + \frac{dz}{dy}dy \quad . \; . \; . \; . \; (1),$$

the coefficients $\frac{dz}{dx}, \frac{dz}{dy}$, being functions of x and y, we have

$$d\;.\;\frac{dz}{dx} = \frac{d^2z}{dx^2}dx + \frac{d^2z}{dx\,dy}dy$$

$$d\;.\;\frac{dz}{dy} = \frac{d^2z}{dy\,dx}dx + \frac{d^2z}{dy^2}dy$$

and, consequently,

$$d^2z = \frac{d^2z}{dx^2}ax^2 + 2\frac{d^2z}{dx\,dy}dx\,dy + \frac{d^2z}{dy^2}dy^2 \quad . \; . \; . \; . \; (2).$$

In like manner, these coefficients being functions of x and y, we have

$$d\;.\;\frac{d^2z}{dx^2} = \frac{d^3z}{dx^3}dx + \frac{d^3z}{dx^2\,dy}dy$$

$$d\;.\;\frac{d^2z}{dx\,dy} = \frac{d^3z}{dx^2\,dy} + \frac{d^3z}{dx\,dy^2}dy$$

$$d\;.\;\frac{d^3z}{dy^2} = \frac{d^3z}{dy^2\,dx} + \frac{d^3z}{dy^3}dy$$

so that

$$d^3z = \frac{d^3z}{dx^3}dx^3 + 3\frac{d^3z}{dx^2\,dy}dx^2\,dy + 3\frac{d^3z}{dx\,dy^2}dx\,dy^2 +$$

$$\frac{d^3z}{dy^3}dy^3 \quad . \; . \; . \; . \; (3),$$

and so on; the numeral coefficients agreeing with those in the corresponding powers of the expanded binomial.

(61.) Having now applied Taylor's theorem to functions of two

variables, we may equally extend Maclaurin's Theorem. For, if in the foregoing development, we suppose x and y each $= 0$, the development will become that of the function F (h, k) according to the powers of h and k; or, substituting x and y for the symbols h and k, since these are indeterminate, we have

$$z = \mathrm{F}(x, y) = [z] + ([\frac{dz}{dx}]\, x + [\frac{dz}{dy}]\, y) + \tfrac{1}{2}([\frac{d^2z}{dx^2}]\, x^2 +$$

$$2\,[\frac{d^2z}{dx\,dy}]\, xy + [\frac{d^2z}{dx^2}]\, y^2) + \&c.$$

The principles by which we have thus extended the theorems of Taylor and Maclaurin are sufficient to enable us to extend these theorems still further, even to the development of functions of any number of variables whatever, but this is unnecessary. It may be remarked, however, that if we wish to develop a function of several variables according to the powers of one of them, it may be done independently of any thing taught in this chapter; for, if all the variables but this one were constants, the development would agree with that already established for functions of a single variable, and, as these constants may take any value whatever, they may obviously be replaced by so many independent variables. We shall give one instance of this extension of Maclaurin's theorem to a function of two independent variables, choosing a form of extensive application and of which the development is known by the name of

Lagrange's Theorem.

(62.) The function which we here propose to develop according to the power of x, is

$$u = \mathrm{F}z, \text{ in which } z = y + xfz,$$

z being obviously a function of the independent variables x and y. We shall first develop $z = y + xfz$ according to the powers of x: this development is by Maclaurin's theorem

$$z = [z] + [\frac{dz}{dx}]\, x + [\frac{d^2z}{dx^2}]\, \frac{x^2}{1 \cdot 2} + [\frac{d^3z}{dx^3}]\, \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.$$

and if we denote according to the notation of Lagrange the successive differential coefficients of fz, relatively to x by $f'z, f''z, f'''z$, &c. we shall have

$$z = y + xfz$$
$$\frac{dz}{dx} = fz + xf'z$$
$$\frac{d^2z}{dx^2} = 2f'z + xf''z = 2\frac{dfz}{dz}\cdot\frac{dz}{dx} + xf''z$$
$$\frac{d^3z}{dx^3} = 3f''z + xf'''z = 3\left(\frac{d^2fz}{dz^2}\cdot\frac{dz^2}{dx^2} + \frac{dfz}{dz}\cdot\frac{d^2z}{dx^2}\right) + xf'''z$$
&c. &c.

Consequently, when $x = 0$,

$$[z] = y$$
$$\left[\frac{dz}{dx}\right] = fy$$
$$\left[\frac{d^2z}{dx^2}\right] = 2\frac{dfy}{dy}fy = \frac{d.(fy)^2}{dy}$$
$$\left[\frac{d^3z}{dx^3}\right] = 3\frac{d^2fy}{dy^2}(fy)^2 + 3\frac{dfy}{dy}\cdot\frac{d.(fy)^2}{dy} = \frac{d^2.(fy)^3}{dy^2} +$$
&c. &c.

Hence

$$z = y + xfz = y + fy\cdot\frac{x}{1} + \frac{d.(fy)^2}{dy}\cdot\frac{x^2}{1\cdot 2} + \frac{d^2.(fy)^3}{dy^2}\cdot\frac{x^3}{1\cdot 2\cdot 3} + \&c. \quad .\ .\ .\ .\ . \quad (1).$$

Now, instead of this development, we should obviously have obtained that of $Fz = F(y + xfz)$, if in place of z and its differential coefficients we had employed Fz and its differential coefficients. We should then have had

$$u = F(y + xfz) \quad .\ .\ \text{therefore}\ .\ . \quad [u] = Fy$$
$$\frac{du}{dx} = \frac{du}{dz}\cdot\frac{dz}{dx} \quad .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \quad \left[\frac{du}{dx}\right] = \frac{dFy}{dy}fy$$
$$\frac{d^2u}{dx^2} = \frac{d^2u}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{du}{dz}\cdot\frac{d^2z}{dx^2} \quad .\ .\ .\ .\ .\ . \quad \left[\frac{d^2u}{dx^2}\right] = \frac{d^2Fy}{dy^2}(fy)^2 +$$
$$\frac{dFy}{dy}\cdot\frac{d.(fy)^2}{dy} = \frac{d.\frac{dFy}{dy}(fy)^2}{dy}$$
&c. &c.

Hence

$$\mathrm{F}z = \mathrm{F}\,(y + xfz) = \mathrm{F}y + \frac{d.\mathrm{F}y}{dy} fy \,. \frac{x}{1} + \frac{d\,.\dfrac{d.\mathrm{F}y}{dy}(fy)^2}{dy} \,. \frac{x^2}{1\cdot 2} + \frac{d^2\,.\dfrac{d.\mathrm{F}y}{dy}(fy)^3}{dy^2} \,. \frac{x^3}{1\cdot 2\cdot 3} + \&\text{c.} \ldots\ldots (2),$$

and this is *Lagrange's Theorem.**

From this remarkable expression, which includes that marked (1), other forms may be readily deduced as particular cases. Of these the two following are the most important.

Put $x = 1$, then the formula (1) becomes

$$z = y + fz = y + fy + \frac{d.(fy)^2}{dy} \,. \frac{1}{1\cdot 2} + \frac{d^2.(fy)^3}{dy^2} \,. \frac{1}{1\cdot 2\cdot 3} + \&\text{c.} \ldots\ldots (3),$$

and the formula (2),

$$u = \mathrm{F}\,(y + fz) = \mathrm{F}y + \frac{d.\mathrm{F}y}{dy} fy + \frac{d.\dfrac{d.\mathrm{F}y}{dy}(fy)^2}{dy} \,. \frac{1}{1\cdot 2} + \frac{d^2.\dfrac{d.\mathrm{F}y}{dy}(fy)^3}{dy^2} \,. \frac{1}{1\cdot 2\cdot 3} + \&\text{c.} \ldots\ldots (4).$$

(63.) We shall terminate the present section with one or two examples of the application of these formulas, referring the student for more ample details on this subject to *Lagrange's Resolution des Equations Numeriques*, note xi.; and *Jephson's Fluxional Calculus*, vol. i.

EXAMPLES.

1. Given $z^3 - qz + r = 0$, to develop z according to the powers of r.

Since here $z = \frac{r}{q} + \frac{z^3}{q}$, we have $y = \frac{r}{q}, fz = \frac{1}{q}z^3 \therefore fy = \frac{1}{q}(\frac{r}{q})^3$

* For another and very complete demonstration of this theorem see note (B) at the end.

$$\therefore \frac{d(fy)^2}{dy} - \frac{1}{q^2} \cdot \frac{d \cdot y^6}{dy} = \frac{6}{q^2} y^5, \ \frac{d^2 (fy)^3}{dy^2} = \frac{1}{q^3} \cdot \frac{d^2 y^9}{dy^2} = \frac{9}{q^3} \cdot \frac{dy^8}{dy} = \frac{8 \cdot 9}{q^3} y^7, \ \&c.$$

Hence, by the formula (3), we have, by putting for y its value $\frac{r}{q}$

$$z = \frac{r}{q} + \frac{1}{q} \cdot \frac{r^3}{q^3} + \frac{6}{1 \cdot 2q^2} \cdot \frac{r^5}{q^5} + \frac{8 \cdot 9}{1 \cdot 2 \cdot 3q^3} \cdot \frac{r^7}{q^7} + \&c.$$

$$= \frac{r}{q} \left(1 + \frac{r^2}{q^3} + \frac{6r^4}{1 \cdot 2q^6} + \frac{8 \cdot 9r^6}{1 \cdot 2 \cdot q^9} + \&c.\right)$$

2. Given the radius vector of an ellipse, viz. (*Anal. Geom.*).

$$r = a \cdot \frac{1 - e^2}{1 + e \cos. \omega}$$

to develop r^n, according to the powers of cos. ω.

Since $r = a\,(1 - e^2) - e \cos. \omega \,.\, r$, we have, by putting y for $a\,(1 - e^2)$ and x for $- e \cos. \omega$,

$$\mathrm{F}r = \mathrm{F}\,(y + xfr) = \mathrm{F}\,(y + x \cdot r) = (y + x \cdot r)^n.$$

Hence, by the formula (2)

$$r^n = y^n + \frac{dy^n}{dy} y \cdot \frac{x}{1} + \frac{d \cdot \frac{dy^n}{dy} y^2}{dy} \cdot \frac{x^2}{1 \cdot 2} + \frac{d^2 \cdot \frac{dy^n}{dy} y^3}{dy^2} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$= y^n + ny^n \cdot \frac{x}{1} + n\,(n + 1)\, y^n \cdot \frac{x^2}{1 \cdot 2} + n\,(n + 1)\,(n + 2)\, y^n \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$= a^n\,(1 - e^2)^n \left(1 - \frac{ne \cos. \omega}{1} + \frac{n\,(n + 1)}{1 \cdot 2} e^2 \cos.^2 \omega - \frac{n\,(n + 1)\,(n + 2)}{1 \cdot 2 \cdot 3} e^3 \cos.^3 \omega + \&c.\right)$$

3. It is required to revert the series

$$\alpha + \beta z + \gamma z^2 + \delta z^3 + \&c. = 0,$$

that is, to express the value of z in terms of the coefficients. Here

$$z = -\frac{\alpha}{\beta} - \frac{z^2}{\beta}(\gamma + \delta z + \&c.) = y + fz$$

therefore, by the formula (3),

$$z = y - \frac{y^2}{\beta^3}(\gamma + \delta y + \&c.) + \frac{d\,.\,\frac{y^4}{\beta^2}(\gamma + \delta y + \&c.)^2}{dy} \cdot \frac{1}{1 \cdot 2} -$$

$$\frac{d^2\,.\,\frac{y^6}{\beta^3}(\gamma + \delta y + \&c.)^3}{dy^2} \cdot \frac{1}{1 \cdot 2 \cdot 3} + \&c.$$

$$= y - \frac{y^2}{\beta}(\gamma + \delta y + \&c.) + 2\frac{y^3}{\beta^2}(\gamma + \delta y + \&c.)^2 +$$

$$\frac{y^4}{\beta^2}(\gamma + \delta y + \&c.)(\delta + 2\varepsilon y + \&c.)$$

$$- 5\frac{y^4}{\beta^3}(\gamma + \delta y + \&c.)^3 + \&c.$$

$$= y - \frac{\gamma}{\beta}y^2 - \frac{\delta}{\beta}y^3 - \frac{\varepsilon}{\beta}y^4 - \&c.$$

$$+ \frac{2\gamma^2}{\beta^2}y^3 + \frac{5\gamma\delta}{\beta^2}y^4 + \&c.$$

$$- \frac{5\gamma^3}{\beta^3}y^4 + \&c.$$

$$+ \&c.$$

where $y = -\frac{\alpha}{\beta}$. consequently

$$z = -\frac{1}{\beta}\alpha - \frac{\gamma}{\beta^3}\alpha^2 + \frac{\delta\beta - 2\gamma^2}{\beta^5}\alpha^3 - \frac{\beta^2\varepsilon - 5\gamma\delta\beta + 5\gamma^3}{\beta^7}\alpha^4 + \&c.$$

4. Given $1 - z + az = 0$ to develop log. z, according to the powers of a.

$$\log. z = a + \tfrac{1}{2}a^2 + \tfrac{1}{3}a^3 + \tfrac{1}{4}a^4 + \&c.*$$

* This we know from other principles; for, since the proposed expression reduces to $z = \frac{1}{1-a}$ $\therefore$ log. $z = -$ log. $(1 - a)$ and this, in the hyperbolic system, is equal to the above series. (*See the Essay on Logarithms*, p. 3.)

CHAPTER VIII.

ON THE MAXIMA AND MINIMA VALUES OF FUNCTIONS OF TWO VARIABLES, AND ON CHANGING THE INDEPENDENT VARIABLE.

(64.) It remains to complete the theory of maxima and minima by applying the principles established in Chapter VI. to functions of two independent variables.

The same character belongs to a maximum or minimum function of two variables that belongs to a maximum or minimum function of one variable, that is, the maximum value exceeds the contiguous values of the function, and the minimum value falls short of them.

Hence, if

$$z = \mathrm{F}\,(x, y)$$

be any function of two variables, which becomes a maximum for certain particular values of them, then h and k being finite increments, however small the condition is that, between such finite values and 0, we must always have

$$\mathrm{F}\,[x, y] > \mathrm{F}\,[x \pm h, y \pm k],$$

and, consequently, (60),

$$(\pm \frac{dz}{dx} h \pm \frac{dz}{dy} k) + \tfrac{1}{2} (\frac{d^2z}{dx^2} h^2 \pm 2 \frac{d^2z}{dxdy} hk + \frac{d^2z}{dy^2} k^2) + \&c. < 0.*$$

If, therefore, of the small values which we suppose h and k to take, h be the smallest, a part of k may be taken so small as to be less than h, or, which is the same thing, equal to one of the values of h between the proposed value and 0, so that we have $h' = k'$; therefore, the above condition is

$$[\pm \frac{dz}{dx} \pm \frac{dz}{dx}]\, h' + \tfrac{1}{2} [\frac{d^2z}{dx^2} \pm 2 \frac{d^2z}{dx\,dy} + \frac{d^2z}{dy^2}]\, h'^2 + \&c. < 0.$$

This condition being similar to (1) art. (49), we infer, by the same reasoning, that

$$\pm \frac{dz}{dx} \pm \frac{dz}{dy} = 0,$$

* This is the manner in which analysts have agreed to express an isolated negative quantity; which must necessarily have resulted from the subtraction of a greater from a less quantity. It is not, however, to be inferred that a negative quantity is less than zero, as the above expression indicates, as such supposition would be manifestly absurd. *Ed.*

which cannot be for both the signs $\pm$ unless

$$\frac{dz}{dx} = 0, \frac{dz}{dy} = 0 \quad . \quad . \quad . \quad . \quad (1).$$

By continuing to imitate the reasoning in (49), we find that these same conditions must exist for all the values of the variables that render the function a *minimum.*

Hence (49), we have, in the case of a ~~minimum~~, the condition

$$\tfrac{1}{2}\left[\frac{d^2z}{dx^2} \pm 2\frac{d^2z}{dx\,dy} + \frac{d^2z}{dy^2}\right] h'^2 + \&c. < 0,$$

and in the case of a minimum,

$$\tfrac{1}{2}\left[\frac{d^2z}{dx^2} \pm 2\frac{d^2z}{dx\,dy} + \frac{d^2z}{dy^2}\right] h'^2 + \&c. > 0,$$

so that, supposing these first terms do not vanish for the values of x and y given by (1), the condition of maximum is

$$\left[\frac{d^2z}{dx^2} \pm 2\frac{d^2z}{dx\,dy} + \frac{d^2z}{dy^2}\right] < 0,$$

and the condition of minimum,

$$\left[\frac{d^2z}{dx^2} \pm 2\frac{d^2z}{dy\,dx} + \frac{d^2z}{dy^2}\right] > 0.$$

In either case, therefore, the expression within the brackets must have the same sign independently of the sign of the middle term. To determine upon what other condition this depends, let us represent the expression by

$$A \pm 2B + C \text{ or } A\left(1 \pm 2\frac{B}{A} + \frac{C}{A}\right).$$

Adding $\frac{B^2}{A^2} - \frac{B^2}{A^2} = 0$ to the quantity within the parenthesis, its form is

$$A\left(\left(1 \pm \frac{B}{A}\right)^2 + \frac{C}{A} - \frac{B^2}{A^2}\right).$$

Now this expression will always have the same sign as A provided C has, and that $\frac{C}{A} > \frac{B^2}{A^2}$, that is, $AC > B^2$ or $AC - B^2 > 0$, because then the factor of A will be necessarily positive. Hence, beside (1), the condition that a maximum or a minimum may exist is

$$\left[\frac{d^2z}{dx^2}\cdot\frac{d^2z}{dy^2}-\left(\frac{d^2z}{dx\,dy}\right)^2\right] > 0 \;\ldots\ldots\; (2),$$

and we are to distinguish the *maximum* from the *minimum* by ascertaining whether the proposed values of x and y render

$$\frac{d^2z}{dx^2} < 0 \text{ or } > 0,$$

or, which amounts to the same, whether

$$\frac{d^2z}{dy^2} < 0 \text{ or } > 0,$$

since $\frac{d^2z}{dx^2}$ and $\frac{d^2z}{dy^2}$ have the same sign.

Should any of the values determined from (1) cause the coefficient of h'^2 to vanish, there will be no maximum or minimum for those values unless the coefficient of the following term vanishes also.

EXAMPLES.

(65.) 1. To determine the shortest distance between two straight lines situated in space.

Let the equations of the two lines be

$$\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\} \text{ and } \left\{\begin{array}{l} x' = a'z' + \alpha' \\ y' = b'z' + \beta' \end{array}\right. \;\ldots\ldots\; (1)$$

then the expression for the distance between any two points (x, y, z), $(x'\ y'\ z')$ is (*Anal Geom.*)

$$\begin{aligned} D^2 = u &= (x - x')^2 + (y - y')^2 + (z - z')^2 \\ &= (\alpha - \alpha' + az - a'z')^2 + (\beta - \beta' + bz - b'z')^2 + (z - z')^2 \end{aligned}$$

and this expression, containing the two independent variables z, z' is to be a minimum. Hence by the condition (1)

$$\left.\begin{array}{l} \frac{du}{dz} = 2(z-z') + 2a(\alpha-\alpha'+az-a'z') + 2b(\beta-\beta'+bz-b'z') = 0 \\ \frac{du}{dz'} = -2(z-z') + 2a'(\alpha-\alpha'+az-a'z') + 2b'(\beta-\beta'+bz-b'z') = 0 \end{array}\right\} .\; (2)$$

and from these equations the proper values of z, z' may be readily determined, which substituted in the expression for u, render it the least possible. That these values really belong to a minimum is evident, because,

$$\frac{d^2u}{dz^2} = 2\,(1 + a^2 + b^2),\ \frac{d^2u}{dz'^2} = 2\,(1 + a'^2 + b'^2),\ \frac{d^2u}{dzdz'}$$
$$= -\,2\,(1 + aa' + bb'),$$

and this proves that $\frac{d^2u}{dz^2}, \frac{d^2u}{dz'^2}$ are both positive, and that

$$\frac{d^2u}{dz^2} \cdot \frac{d^2u}{dz'^2} - (\frac{d^2z}{dxdy})^2 \mathrel{\not\,} 7\ 0.$$

Since the equations of a straight line passing through two points (x, y, z), (x', y', z') are

$$\left.\begin{array}{l} x - x' = a''\,(z - z') \\ y - y' = b''\,(z - z') \end{array}\right\}$$

we have, by substitution, when these points are on the lines (1)

$$\left.\begin{array}{l} \alpha - \alpha' + az - a'\,z' = a''\,(z - z') \\ \beta - \beta' + bz - b'\,z' = b''\,(z - z') \end{array}\right\} \ldots\ldots (3)$$

hence, if this line be that in question, we have, by combining the equations (2) with these, the conditions

$$1 + aa'' + bb'' = 0,\ 1 + a'\,a'' + b'\,b'' = 0,$$

which conditions shew, that this *minimum* straight line is perpendicular to *both* the lines (1). (See *Anal. Geom.*) From these conditions we get

$$a'' = \frac{b' - b}{a'b - ab'},\ b'' = -\ \frac{a' - a}{a'b - ab'}.$$

by means of which, and the equations (3), the expiession for D becomes

$$D = \frac{(z - z')}{a'b - ab'} \sqrt{(a - a')^2 + (b - b')^2 + (a'b - ab')^2}$$

in which, if we substitute the value of $z - z'$ deduced from (2), we obtain, finally,

$$D = \frac{(b - b')\,(\alpha - \alpha') - (a - a')\,(\beta - \beta')}{\sqrt{(a - a')^2 + (b - b')^2 + (a'b - ab')^2}}.$$

If the numerator of this expression vanish, we shall have $D = 0$; so that, in this case, the lines will intersect. Indeed, the condition of intersection of the two lines, (1), we know (*Anal. Geom.*) to be

$$(b - b')\,(\alpha - \alpha') = (a - a')\,(\beta - \beta').$$

2. Among all rectangular prisms to determine that which, having a given volume, shall have the least possible surface.

Representing the three contiguous edges of the prism by x, y, z, and the volume by a^3 we have

$$u = 2xy + 2xz + 2yz = \text{minimum}.$$

but since

$$xyz = a^3 \therefore z = \frac{a^3}{xy}$$

$$\therefore u = 2xy + \frac{2a^3}{y} + \frac{2a^3}{x} = \text{minimum}.$$

therefore we must have the conditions

$$\frac{du}{dx} = 0, \frac{du}{dy} = 0$$

that is,

$$2y = \frac{2a^3}{x^2} = 0, 2x - \frac{2a^3}{y^2} = 0$$

from which we obtain

$$y = x = a \therefore z = a.$$

If these values really correspond to a minimum they must fulfil the conditions

$$[\frac{d^2u}{dx^2}] > 0, [\frac{d^2u}{dy^2}] > 0, [\frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} - (\frac{d^2u}{dx\,dy})^2] > 0,$$

and these conditions are fulfilled, since

$$[\frac{d^2u}{dx^2}] = 4, [\frac{d^2u}{dy^2}] = 4, [\frac{d^2u}{dx\,dy}] = 2.$$

Hence the required prism must be a cube.

In the preceding example we might have concluded, without recurring to these conditions, that the results obtained belong to the required minimum, there being obviously no other maximum or minimum, except that which belongs to $x = 0$, $y = 0$, $z = \infty$, these being the values which cause the differential coefficients to become *infinite*, (see *art.* 50.)

3. To divide a given number, a, into three parts, such that the continued product of the mth power of the first part, the nth power of the second part, and the pth power of the third, may be the greatest possible.

The three parts are $\frac{ma}{m+n+p}$, $\frac{na}{m+n+p}$, $\frac{pa}{m+n+p}$ so that the three parts are to each other as the exponents of the proposed powers.

4. To determine the greatest triangle that can be enclosed by a given perimeter.

The triangle must be equilateral.*

On changing the independent variable.

(66.) It is frequently requisite to employ the differential coefficients $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ &c., in which x is considered as the principal variable under a change of hypothesis, x, and consequently y being assumed as a function of some new variable t.

It is therefore of consequence to ascertain what changes take place in the expressions for these coefficients in such cases. This we may do as follows:

Since according to the new hypothesis

$$y = \mathrm{F}x \text{ and } x = ft$$

therefore (33)

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \therefore \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{(dy)}{(dx)},$$

where for brevity (dy) is put for $\frac{dy}{dt}$ and (dx) for $\frac{dx}{dt}$.

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} \cdot \frac{dx^2}{dt^2} + \frac{dy}{dx} \cdot \frac{d^2x}{dt^2}$$

$$\therefore \frac{d^2y}{dx^2} = (\frac{d^2y}{dt^2} - \frac{(dy)}{(dx)} \cdot (d^2x) \div (dx)^2 = \frac{(d^2y)(dx) - (d^2x)(dy)}{(dx)^3}.$$

In a similar manner we might, if necessary, find the expression for $\frac{d^3y}{dx^3}$. It appears, therefore, that

$$\frac{dy}{dx} = \frac{(dy)}{(dx)}$$

* For more examples the student may refer to *Jephson's Fluxional Calculus*, to *Garnier's Calcul Differentiel*, or to *Puissant's Problèmes de Géométric.*

$$\frac{d^2y}{dx^2} = \frac{(d^2y)\,(dx) - (d^2x)\,(dy)}{(dx)^3}$$

&c. &c.

If $t = y$ the hypothesis requires that y be considered as the principal and x as the dependent variable. In this case

$$(dy) = \frac{dy}{dy} = 1,\ (d^2y) = 0,\ \&c.\ (dx) = \frac{dx}{dy},\ (d^2x) = -\frac{d^2x}{dy^2},\ \&c.$$

$$\therefore \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{(dx)}$$

$$\frac{d^2y}{dx^2} = -\frac{(d^2x)}{(dx)^2}$$

&c. &c.

These formulas will be brought into use in the second section.

CHAPTER IX.

ON THE CASES IN WHICH TAYLOR'S THEOREM FAILS.

(67.) It has been shown, in Chapter II., that the general development of the function F $(x + h)$ always proceeds according to the ascending positive powers of h, and the principle upon which this fact has been established is this: viz. that Fx and F $(x + h)$ must necessarily contain the same number of values; or in other words, the same radicals that enter Fx must also enter F$(x+h)$ but no others. Hence we might extend the proposition established in (4), and say, that not only the general development proceeds according to the increasing positive powers of h, but also every particular development, provided the particular value F $(a + h)$ contain the same radicals as Fa, and no more; and provided, moreover, that Fa is not infinite, which we have seen it must be for the true development of F $(a+h)$ to contain a negative power of h, or a log. h, a cot. h, &c.

(68.) As F $(x + h)$ must contain the same radicals as Fx and no

others, it follows that $F(x+h) - Fx$, must contain the same radicals as Fx. Now, as multiplying or dividing an expression by any *rational* quantity, can neither introduce nor destroy radicals in that expression, we infer that the expression for $\frac{F(x+h) - Fx}{h}$, h being rational, must contain the very same radicals as Fx, and no others, whatever be the value of h; but when $h = 0$

$$\frac{F(x+h) - Fx}{h} = \frac{dFx}{dx},$$

hence the first differential coefficient must contain the same radicals as the function Fx, and, by the same reasoning, the second differential coefficient must contain the same radicals as the first; consequently the same radicals must enter each differential coefficient that enter into the original function. If, therefore, there be given to x such a particular value a, that any one of the expressions that may be under radicals in Fx may become 0, that radical will of course vanish from the function, and consequently from its differential coefficients. But if $a + h$ be substituted for x instead of a, the same radical will necessarily be preserved in $F(a+h)$, although it will still vanish from Fa and the differential coefficients. It follows, therefore, that $F(a+h)$ will have more values than

$$Fa + [\frac{dFx}{dx}]\,h + [\frac{d^2Fx}{dx}]\,\frac{h^2}{1 \cdot 2} + \&c.$$

so that this cannot be the true development of $F(a+h)$. It is easy to explain why, in such cases as this, one of the coefficients and indeed all that follow this, must become infinite, for $x = a$. For the exponent of the radical which vanishes for this value, is diminished by unity at each differentiation, and being fractional the expression under it will at length appear with a *negative* exponent, and will continue to have a negative exponent in all the succeeding coefficients;* so that these, when a is put for x, become infinite. We have observed above, that the failing cases of which we are speaking,

* This does not invalidate the previous assertion, that the *same* radicals enter the coefficients, that appear in the original function; for the radical still remains, however we increase or diminish the fractional exponent by integers, for

$$fx^{\frac{1}{n} \pm p} = \sqrt[n]{fx^{1 \pm np}}.$$

arise from the circumstance of $x = a$ causing an expression to disappear, which is under a radical in Fx. The student must not confound this disappearance of a radical, with that which may arise from a *factor* by which it is multiplied becoming 0 for $x = a$, for though a radical may disappear in this way from Fx, it will not disappear from *all* the differential coefficients, and therefore Taylor's development will hold. Thus if a radical in Fx is multiplied by $(x - a)^m$, m being a positive whole number, this radical will disappear when $x = a$, but in the mth differential coefficient the factor will be $(x - a)^{m-m}$, which does not vanish when $x = a$, but becomes $= 1$, and thus the radical with which it is connected will appear in this coefficient.

We conclude, therefore, that there are but two classes of values for which Taylor's development fails; 1° those which, put for x, render $\mathrm{F}x = \infty$; that is, those which are roots of the equation $\dfrac{1}{\mathrm{F}x} = 0$: and 2° those which substituted for x, cause an expression under a radical, to vanish from Fx, and not from F $(x + h)$. To this latter class belongs the value $x = a$ for every function containing $\sqrt{x - a}$, for the value $x = a$ causes this radical to vanish from Fx, but in F $(x + h)$ it enters as $\sqrt{h}$.

(69.) In order to examine these cases more completely, let in general

$$\mathrm{F}(a+h) = \mathrm{A} + \mathrm{B}h + \mathrm{C}h^2 + \mathrm{D}h^3 + \ldots . \mathrm{M}h^n + \mathrm{N}h^{n+\frac{1}{p}} + \&c. \quad (1)$$

represent the true development of F $(x + h)$ for $x = a$, in which $n + \dfrac{1}{p}$ denotes a fraction falling between the numbers n and $n + 1$. We shall show that the $n + 1$th differential coefficient derived from the function Fx becomes infinite for $x = a$, as also all that follow this, but the preceding coefficients are all finite.

Since in the development (1), h has no fixed value, we may differentiate relatively to h, and we shall have,

$$\frac{d\mathrm{F}(a+h)}{dh} = \mathrm{B} + 2\mathrm{C}h + 3\mathrm{D}h^2 + \ldots . + \mathrm{M}_1{}^* h^{n-1} + \mathrm{N}_1 h^{n+\frac{1}{p}-1} + \&c.$$

$$\frac{d^2\mathrm{F}(a+h)}{dh^2} = 2\mathrm{C} + 2 \cdot 3\mathrm{D}h + \ldots . + \mathrm{M}_2\, h^{n-2} + \mathrm{N}_2 h^{n+\frac{1}{p}-1} + \&c.$$

&c. &c.

* For brevity M_n will be here used to denote the coefficient of h^{n-1} in the nth differential coefficient derived from $\mathrm{M}h^n$.

Now, by (30), these several differential coefficients are the same as

$$\left[\frac{d\mathrm{F}(x+h)}{dx}\right], \left[\frac{d^2\mathrm{F}\,(x+h)}{dx^2}\right], \text{\&c.}$$

the brackets denoting the values when $x = a$.
Hence, by substitution, in the foregoing equations,

$$\left[\frac{d\mathrm{F}\,(x+h)}{dx}\right] = \mathrm{B} + 2\mathrm{C}h + 3\mathrm{D}h^2 + \ldots .\ \mathrm{M}_1 h^{n-1} + \mathrm{N}_1 h^{n+\frac{1}{p}-1}$$

$$+ \text{\&c.} \ldots .\ (2)$$

$$\left[\frac{d^2\mathrm{F}\,(x+h)}{dx^2}\right] = 2\mathrm{C} + 2 \cdot 3\mathrm{D}h + \ldots \ldots .\ \mathrm{M}_2 h^{n-2} + \mathrm{N}_2 h^{n+\frac{1}{p}-2}$$

$$+ \text{\&c.} \ldots .\ (3)$$

&c. &c.

Putting, now, $h = 0$ in the equations (1), (2), (3), &c. we have the following results.

$$\mathrm{F}a = \mathrm{A}$$

$$\left[\frac{d\mathrm{F}x}{dx}\right] = \mathrm{B}$$

$$\left[\frac{d^2\mathrm{F}x}{dx^2}\right] = 2\mathrm{C}$$

$$\left[\frac{d^3\mathrm{F}x}{dx^3}\right] = 2\,.\,3\ \mathrm{D}$$

$$\vdots$$

$$\left[\frac{d^n\mathrm{F}x}{dx^n}\right] = \mathrm{M}_n h^{n-n} + \mathrm{N}_n 0^{n-n+\frac{1}{p}} + \text{\&c.} = \mathrm{M}_n$$

$$\left[\frac{d^{n+1}\mathrm{F}x}{dx^{n+1}}\right] = \mathrm{N}_{n+1} 0^{\frac{1}{p}-1} + \text{\&c.} = \frac{\mathrm{N}_{n+1}}{0} = \infty,$$

&c. &c.

all the succeeding differential coefficients being obviously infinite, because the exponent $\frac{1}{p} - 1$, which is already negative, continually diminishes by unity.

(70.) It follows, therefore, that if the true development of $\mathrm{F}(x+h)$, arranged according to the increasing exponents of h, contain for $x = a$ a fractional power of h, comprised between the powers h^n and

h^{n+1}, then the several terms of this development will be correctly determined by Taylor's theorem, as far as the term containing h^n inclusively; but the terms beyond this become infinite, and therefore do not belong to the true development.

If a term, in the true development of $F(a+h)$, contain a negative power of h, this should be the leading term, as the arrangement is according to the increasing exponents; therefore, this first term, when $x=a$ and $h=0$, must be infinite, and consequently all the differential coefficients, (2), (3), &c. must be infinite.

(71.) The converse of these inferences are true, viz. 1°. If, in the general development of $F(x+h)$, the coefficient $\frac{d^{n+1}Fx}{dx^{n+1}}$ is the first which becomes infinite for a particular value of x, then, in the true development, arranged according to the increasing exponents of h, the term immediately succeeding that which contains h^n, will contain a fractional power of h, the exponent being between n and $n+1$. For it is obvious, that in order that the $n+1$th may be the first of the coefficients (2), (3), &c. which contain a negative power of h, $h^{n+\frac{1}{p}}$ must be the first fractional power of h which enters the development (1), $n+\frac{1}{p}$ being between n and $n+1$. 2° If, for a particular value of x, the function Fx become infinite then will all the differential coefficients become also infinite, and the true development will contain a negative power of h, or else a log. h, a cot. h, &c. For, if the true development of $F(a+h)$ did not contain a negative power of h, nor a log. h, a cot. h, &c. Fa, which this becomes when $h=0$, could not be infinite; hence, such a function of h must enter, and therefore, as shown above, all the differential coefficients become infinite, for $x=a$. It is, therefore, necessary to examine the function Fa, before we deduce the coefficients from Fx.

(72.) To obtain the true development of the function for those particular values of the variable, which cause Taylor's theorem to fail, the usual course is to recur to the ordinary process of common algebra, after having substituted $a+h$ for x in Fx.

Suppose, for example, the function were

$$Fx = 2ax - x^2 + a\sqrt{x^2 - a^2},$$

and that we required the development of $F(x+h)$, for $x=a$.

Taking the differential coefficient, we have

$$\frac{dy}{dx} = 2(a - x) + \frac{ax}{\sqrt{x^2 - a^2}} \therefore \left[\frac{dy}{dx}\right] = \infty,$$

&c. &c.

As, therefore, the first differential coefficient becomes infinite for the proposed value of x, we conclude that the true development of the function for that value, when arranged according to the increasing exponents of h, has a fractional power of h in the second term, the exponent of this power being between 0 and 1.

Substituting, then, $a + h$ for x in Fx, we have

$$F(a + h) = a^2 - h^2 + a\sqrt{2ah + h^2}$$
$$= a^2 - h^2 + ah^{\frac{1}{2}}(2a + h)^{\frac{1}{2}}$$

Developing $(2a + h)^{\frac{1}{2}}$ by the binomial theorem, we have

$$(2a + h)^{\frac{1}{2}} = (2a)^{\frac{1}{2}} + \frac{h}{2(2a)^{\frac{1}{2}}} - \frac{h^2}{8(2a)^{\frac{3}{2}}} + \&c.$$

consequently,

$$F(a + h) = a^2 + a(2a)^{\frac{1}{2}}h^{\frac{1}{2}} + \frac{h^{\frac{3}{2}}}{2(2a)^{\frac{1}{2}}} - h^2 - \frac{h^{\frac{5}{2}}}{8(2a)^{\frac{3}{2}}} + \&c.$$

Again let the function be

$$Fx = \sqrt{x} + (x - a)^2 \log.(x - a),$$

and let it be required to find the development of $F(a + h)$. Here

$$Fa = \sqrt{a} + 0 \times \infty.$$

It becomes necessary, therefore, first to ascertain whether this expression is infinite; for, if it be, we are not to proceed with the differentiation, but to infer, agreeably to art. (70), that the proposed function, and all the differential coefficients, become infinite for $x = a$ and that consequently the true development contains either a negative power of h, or a logarithm of h. Now, by the method explained in (44), we find that, when $x = a$, the true value of

$$(x - a)^2 \log.(x - a) = \frac{(x - a)^2}{\dfrac{1}{\log.(x - a)}}$$

is infinite. Hence the development of $F(a + h)$ contains $\log. h$, for

we readily see that no negative power of h can enter. Substituting $a + h$ for x, in Fx, we have

$$F(a+h) = (a+h)^{\frac{1}{2}} + h^2 \log. h.*$$

SCHOLIUM.

(73.) In the preceding remarks on the development of functions for particular values of the variable, we have said nothing about the values of h, the increment of that variable, having indeed considered that increment as indeterminate, or rather of arbitrary value. It must, however, be observed that, although the particular value which we give to x does not, in any case, fix the value of h, it may nevertheless fix the limit between which and 0 all the values given to h must be comprised, in order that for particular values of x, Taylor's development may not fail. This fact is very plain, for if the development holds for all values of x from $x = a$ up to $x = b$, but fails for

* The above example is from *Lagrange*, (*Calcul des Fonctions*, p. 75,) who has given a faulty solution of it, which however is copied by *Garnier* and other writers on the Calculus. The solution here objected to is as follows:

"Soit

$$fx = \sqrt{x} + (x-a)^2 \log. (x-a)$$

on aura ces fonctions dérivées

$$f'x = \frac{1}{2\sqrt{x}} + 2(x-a)\log.(x-a) + x - a$$

$$f''x = -\frac{1}{4x\sqrt{x}} + 2\log.(x-a) + 3$$

$$f''' = \frac{3}{8x^2\sqrt{x}} + \frac{2}{x-a}$$

&c.

Si on fait $x = a$, la fonction seconde $f''x$ devient infine, ainsi que toutes les suivantes.

"Ainsi le développment de $f(x+h)$ par la formule générale deviendra fautif dans le cas de $x = a$, *et il contiendra nécessairement le terme* $h^2 \log. h$."

This solution is faulty, inasmuch as it is assumed that $f''x$ is the first derived function that becomes infinite for $x = a$, whereas $f'x$ and fx are also infinite; but a greater fault is, that this process does not lead to the true conclusion, for the inference in italics does not follow from it, but this, viz. that the sought development contains neither log. of h nor a negative power of h, but it contains a fractional power of h, the exponent being between 1 and 2, which is not the true conclusion.

$x = b$, then will the development hold when $a + h$ is substituted for x in Fx, provided h be taken between the limits $h = 0$ and $h = b - a$, or more strictly, provided it does not exceed these limits. In like manner, if the development hold for all values of x from $x = b$ down to $x = a$, but fails for $x = a$, then will the development hold when $b - h$ is substituted for x for all values of h from $h = 0$, to $h = b - a$, but it will not hold for the value of h immediately succeeding this last; and it is obvious that h will always be subject to such restrictions unless the development holds, not merely for $x = a$, but universally. When, therefore, we find that for $x = a$ the differential coefficients do not any of them become infinite, all that we can conclude is that the development of $F(a \pm h)$ is according to Taylor's theorem for all values of h between some certain finite value h', which may indeed be indefinitely small, and 0, and it is only when this is not the case that the theorem is said by analysts to *fail*. We have thought it necessary to point out these circumstances to the student, seeing that some authors, from not attending to them, have fallen into very important errors, and have laid down erroneous doctrines with respect to the failing cases of Taylor's theorem. Thus Mr. Jephson at page 191, vol. i. of his Fluxionial Calculus, a work containing much valuable information, says "It may further be observed that Taylor's theorem always fails when the assigned value of x causes any of the terms to become imaginary, and that this may take place without causing the function itself to be imaginary; thus take $fx = c + x^2 \sqrt{x - a}$ if we suppose $x = 0, fx = c, f'x = 0$, but $f''x, f'''x$ . . . . all contain $\sqrt{-a}$." From this it would appear that Taylor's theorem may fail to give the true development in other cases besides those which cause the differential coefficients to become infinite, which, however, is not true. Whenever, for any particular value of x, Taylor's coefficients become imaginary, we must infer, agreeably to the statement in (4), that the function $F(x + h)$ becomes imaginary for that value of x; h being of course limited as above explained. In the example just quoted, where

$$f'x = 2x \sqrt{x - a} + \frac{\frac{1}{2}x^2}{\sqrt{x - a}}$$

$$f''x = 2 \sqrt{x - a} + \frac{2x}{\sqrt{x - a}} - \frac{\frac{1}{4}x^2}{(x - a)^{\frac{3}{2}}}$$

&c. &c.

the function $f(x + h)$ becomes, when $x = 0, f(0 + h) = c + h^2 \sqrt{h - a}$ and the development is

$$c + h^2 \sqrt{h - a} = c + 0h + \sqrt{-a}\, h^2 + \frac{1}{2\sqrt{-a}} h^3 + \&c.$$

and this is the true development, for h must not exceed the limits $h = 0$, and $h = a$, since $x = a$ causes the differential coefficients to become infinite, and therefore the development to fail.

With regard to the failing cases of Maclaurin's theorem, it may be observed that they are very different from the failing cases of Taylor's. Whatever be the form of the proposed function, its *general* development, according to Taylor's theorem, never fails; but the failure of Maclaurin's theorem always arises from the *form* of the proposed function and it is the *general* development that fails, and consequently all the particular cases. For it is obvious that every function or any of its differential coefficients which become infinite when $x = 0$, will fail to be developable by Maclaurin's theorem.

Before terminating these remarks it may be proper to observe that the student is not to attribute what analysts have been pleased to term the *failing cases* of these theorems to any defect in the theorems themselves; on the contrary they would be very defective if they did not exhibit such cases. All that is meant is, that the function in *particular states* may fail to be developable according to Taylor's series, and under *particular forms* it may fail to be developable according to Maclaurin's series; so that, in fact, these theorems fail to give the true development only when that development is impossible.

(74.) Let us now examine implicit functions, and let us suppose that $x = a$ causes a radical to vanish from Fx in consequence of a factor of it vanishing; we have seen (68) that such radical will reappear in some of the differential coefficients, suppose it appears in the first which requires that the factor spoken of be $x - a$, then for $x = a$ this coefficient will have more values than the proposed function, as it contains a radical more. But if the function that Fx or y is of x be only implicitly given, that is by means of an equation without radicals, we know (52) that the expression for $\frac{dy}{dx}$ will be also without radicals, and from such an expression it does not at first seem clear how

we are to deduce the multiple values alluded to, and which might be obtained by solving the equation for y and thus introducing the radical. But since the expression for $\frac{dy}{dx}$ appears under the form of a fraction, viz. (52)

$$\frac{dy}{dx} = -\frac{\frac{du}{dx}}{\frac{du}{dy}} \quad . \quad . \quad . \quad . \quad (1),$$

we readily perceive that one case is possible, and only one, in which this fraction may take a multiplicity of values besides those *implied* in y, viz. the case in which it becomes $\frac{0}{0}$; that is, when the following conditions exist simultaneously, viz.

$$\frac{du}{dx} = 0, \frac{du}{dy} = 0 \quad . \quad . \quad . \quad . \quad (2);$$

so that these conditions are those which must exist for every value of x which destroys a radical in Fx but not in $\frac{dy}{dx}$.

(75.) Whenever, therefore, any particular value of x destroys a radical in y, but not in $\frac{dy}{dx}$, then the expression (1) must take the form $\frac{0}{0}$, and admit of the necessary multiple values.

The rules laid down in (41) enable us to determine the true value of the fraction (1) in the proposed circumstances, that is when it takes the form $\frac{0}{0}$, for there is but one independent variable, viz. x. By differentiating numerator and denominator separately, we have, by the article referred to,

$$[\frac{dy}{dx}] = -[\frac{\frac{d^2u}{dx^2} + \frac{d^2u}{dx\,dy} \cdot \frac{dy}{dx}}{\frac{d^2u}{dx\,dy} + \frac{d^2u}{dy^2} \cdot \frac{dy}{dx}}] \quad . \quad . \quad . \quad . \quad (3);$$

hence,

$$[\frac{d^2u}{dx^2} + 2\frac{d^2u}{dx\,dy} \cdot \frac{dy}{dx} + \frac{d^2u}{dy^2} (\frac{dy}{dx})^2] = 0.$$

This being a quadratic equation furnishes *two* values for $[\frac{dy}{dx}]$,

which are all that belong to the fraction (1) or to $[\frac{dy}{dx}]$ unless, indeed, both numerator and denominator of (3) also vanish for a value of x given by the conditions (2), in which case we must differentiate again as in art. (41), when the value of $[\frac{dy}{dx}]$ will be given by an equation of the third degree which will furnish *three* values, and so on; and in general, if the fraction or $[\frac{dy}{dx}]$ admit of n values, the equation which determines them can be obtained only by differentiating n times, which will lead to an equation of the nth degree, and it is plain that the radical destroyed in Fx must be of the $\frac{1}{n}$th degree, seeing that it gives to $\frac{dy}{dx}$ n additional values.

Suppose, for example, we had the function

$$y = x + (x - a)\sqrt{x - b}$$

$$\therefore \frac{dy}{dx} = 1 + \sqrt{x - b} + \frac{x - a}{2\sqrt{x - b}}$$

and for $x = a$

$$[y] = a,\ [\frac{dy}{dx}] = 1 \pm \sqrt{a - b}$$

so that the radical which disappears in y appears in $\frac{dy}{dx}$; this, therefore, has *two* values.

Now let the same function be given in an implicit form and without radicals, viz.

$$u = (y - x)^2 - (x - a)^2 (x - b) = 0,$$

$$\therefore \frac{du}{dx} = -2(y - x) - (x - a)(3x - 2b + a),\ \frac{du}{dy} = 2(y - x).$$

Since for $x = a$, $y = a$, therefore both these expressions become 0; hence $[\frac{dy}{dx}] = \frac{0}{0}$. Taking, then, the differentials of both numerator and denominator of the fraction,

$$\frac{2(y - x) + (x - a)(3x - 2b + a)}{2(y - x)} = \frac{0}{0},$$

we find when $x = a$ that it becomes

$$\left[\frac{dy}{dx}\right] = 1 + \frac{a - b}{\left[\frac{dy}{dx}\right] - 1},$$

therefore,

$$\left[\frac{dy}{dx} - 1\right]^2 - (a - b) = 0 \therefore \left[\frac{dy}{dx}\right] = 1 \pm \sqrt{a - b},$$

as before.

(76.) Suppose now that the radical which disappears from Fx by reason of a factor, disappears also from $\frac{dy}{dx}$, and that it appears in $\frac{d^2y}{dx^2}$, which is the same as supposing that the vanishing factor is $(x - a)^2$. In this case $\frac{dy}{dx}$ will have the same number of values that Fx has for $x = a$, but additional values will belong to $\frac{d^2y}{dx^2}$, so that we must have

$$\left[\frac{d^2y}{dx^2}\right] = \frac{0}{0}$$

and the true values, when the function is implicit, will all be determined by the principles already employed.

For example: the explicit function

$$y = x + (x - a)^2 \sqrt{x}$$

gives when $x = a$

$$[y] = a, \left[\frac{dy}{dx}\right] = 1, \left[\frac{d^2y}{dx^2}\right] = \pm 2 \sqrt{a},$$

and we shall see that the same values are equally given by the implicit function

$$(y - x)^2 = (x - a)^4 x,$$

by applying the foregoing process to $\frac{d^2y}{dx^2}$.

For by successively differentiating, and representing, for brevity, the several coefficients derived from y by p', p'', &c. we have

$$2(p' - 1)(y - x) = (x - a)^3 (5x - a) \therefore [p'] = 1$$
$$(p' - 1)^2 + p''(y - x) = 2(x - a)^2 (5x - 2a)$$

$$\therefore [p''] = [\frac{2(x-a)^2(5x-2a)-(p'-1)^2}{y-x}] = \frac{0}{0}$$

$$= [\frac{6(x-a)(5x-3a)-2p''(p'-1)}{p'-1}] = \frac{0}{0}$$

$$= [\frac{60x-48a-2p'''(y'-1)-2p''^2}{p''}] = \frac{12a-2[p''^2]}{[p'']}$$

$$\therefore 3[p''^2] = 12a \therefore [p''] = \pm 2\sqrt{a},$$

as before.

The two examples following will suffice to exercise the student in this doctrine, which is merely an extension of the principles treated in Chapter V. to implicit functions.

1. Given

$$y^3 = (x-a)^3(x-b)$$

to determine the values of $\frac{dy}{dx}$, when $x = a$,

$$[\frac{dy}{dx}] = \sqrt[3]{a-b}.$$

2. Given

$$(y-x)^2 - (x-a)^4(x-b) = 0$$

to determine the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, when $x = a$.

$$[\frac{dy}{dx}] = 1, [\frac{d^2y}{dy^2}] = 2\sqrt{a-b}.$$

We here terminate the First Section, having fully considered the various particulars relating to the differentiation of functions in general.

SECTION II.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE THEORY OF PLANE CURVES.

CHAPTER I.

ON THE METHOD OF TANGENTS.

(77.) We now proceed to apply the Calculus to Geometry, and shall first explain the method of drawing tangents to curves.

The general equation of a secant passing through two points (x', y'), (x'', y''), in any plane curve, is (*Anal. Geom.*)

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$

$y' - y''$, being the increment of the ordinate or proposed function corresponding to $x' - x''$ the increment of the abscissa or independent variable. The *limit* of the ratio of these increments, by the principles of the calculus, is $\frac{dy'}{dx'}$; that is to say, such is the representation of the ratio when $x' - x'' = 0$, and, consequently, $y' - y'' = 0$. But when this is the case the secant becomes a tangent. Hence *the equation of the tangent, through any point (x', y') of a plane curve, is*

$$y - y' = \frac{dy'}{dx'}(x - x') \quad . \quad . \quad . \quad . \quad (1).$$

It appears, therefore, that the differential coefficient $\frac{dy'}{dx'}$ for any proposed point in the curve has for its value the trigonometrical tangent of the angle included by the linear tangent and the axis of x, that is, provided the axes are rectangular. If the axes are oblique, the same coefficient represents the ratio of the sines of the inclinations of the linear tangent to these axes. (*See Anal. Geom.*)

By means of the general equation (1) we can always readily determine the equation of the tangent to any proposed plane curve when the equation of the curve is given, nothing more being required than to determine from that equation the differential coefficient.

Suppose, for example, it were required to find the particular form for the ellipse. We here have to determine $\frac{dy'}{dx'}$ from the equation

$$Ay'^2 + Bx'^2 = A^2B^2,$$

and which is

$$\frac{dy'}{dx'} = -\frac{B^2x'}{A^2y'}$$

therefore the equation of the tangent is

$$y - y' = -\frac{B^2x'}{A^2y'}(x - x'),$$

(x', y') being any point in the curve, and (x, y) any point in the tangent.

Again; let it be required to determine the general equation of the tangent to a line of the second order.

By differentiating the general equation

$$Ay'^2 + Bx'y' + Cx'^2 + Dy' + Ex' + F = 0, *$$

we have

$$2Ay'\frac{dy'}{dx'} + Bx'\frac{dy'}{dx'} + By' + 2Cx' + D\frac{dy'}{dx'} + E = 0,$$

$$\therefore \frac{dy'}{dx'} = -\frac{2Cx' + By' + E}{2Ay' + Bx' + D}$$

so that the general equation is

$$y - y' = -\frac{2Cx' + By' + E}{2Ay' + Bx' + D}(x - x').$$

(78.) As the normal is always perpendicular to the tangent, its general equation must be, from (1),

$$y - y' = -\frac{1}{\frac{dy'}{dx'}}(x - x') \dagger \ldots . (2).$$

* The general equation of lines of the second order in its most commodious form is

$$y'^2 = mx' + nx'^2,$$

from which, by differentiation, we have

$$\frac{dy'}{dx'} = \frac{m + 2nx'}{2y'}$$

and the equation of the tangent to a line of the second order is therefore

$$y - y' = \frac{m + 2nx'}{2y'}(x - x').$$

† $= -\frac{dx'}{dy'}(x - x').$

ED.

It is easy now to deduce the expressions for the subtangent and subnormal. For if, in the equation of the tangent, we put $y = 0$, the resulting expression for $x - x'$ will be the analytical value of that part of the axis of x intercepted between the tangent and the ordinate y' of the point of contact, that is to say, it will be the value of the subtangent $T_{,}$ (*Anal. Geom.*),

$$\therefore T_{,} = -\frac{y'}{\frac{dy'}{dx'}} \quad . \; . \; . \; . \; (3).$$

If, instead of the equation of the tangent, we put $y = 0$ in that of the normal, then the resulting expression for $x - x'$ will be the value of the intercept between the normal and the ordinate y', that is, it will belong to the subnormal $N_{,}$,

$$\therefore N_{,} = y' \frac{dy'}{dx'} \quad . \; . \; . \; . \; (4).$$

As to the length of the tangent T, since $T = \sqrt{y'^2 + T_{,}^2}$, we have, in virtue of (3),

$$T = y' \sqrt{\left(1 + \frac{1}{\frac{dy'^2}{dx'^2}}\right)} \quad . \; . \; . \; . \; (5).$$

Also, since the length of the normal N is $N = \sqrt{y'^2 + N_{,}^2}$ we have, by (4),

$$N = y' \sqrt{\left(1 + \frac{dy'^2}{dx'^2}\right)} \quad . \; . \; . \; . \; (6).$$

The foregoing expressions evidently apply to any plane curve whatever, that is, to any curve that may be represented by an equation between two variables, whatever be its degree, or however complicated its form.

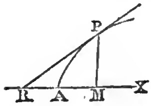
We shall now give a few examples principally illustrative of the method of drawing tangents to the higher curves, for which purpose we shall obviously require only the formula (3), for it is plain that to any point in a curve we may at once draw a tangent, when the length and position of the subtangent is determined. Or, knowing the point (x', y'), we may, by putting successively $x = 0$ and $y = 0$, in the equation (1), determine the two points in which the required tangent ought to cut the axes of the coordinates and then draw it through them. If $\frac{dy'}{dx'}$

is 0 at the proposed point, the tangent will be parallel to the axes of x, because, as remarked above, $\frac{dy'}{dx'}$ is the value of the trigonometrical tangent of the inclination of the linear tangent to the axes of x, and for this reason also the tangent will be parallel to the axes of y when $\frac{dy'}{dx'}$, is infinite at the proposed point.

EXAMPLES.

(79.) 1. To draw a tangent to a given point P in the common or conical parabola.

By the equation of the curve



$$y^2 = px$$

$$\therefore \frac{dy'}{dx'} = \frac{p}{2y'}$$

$$\therefore \mathrm{T}_{,} = y' \div \frac{p}{2y'} = \frac{2y'^2}{p} = 2x'.$$

Hence, having drawn from P, the perpendicular ordinate PM, if we set off the length, MR, on the axis of x, equal to twice AM, and then draw the line RP, it will be the tangent required.

2. To determine the subtangent and subnormal at a given point (x', y') in the parabola of the nth order, represented by the equation

$$y = ax^n$$

$$\therefore \frac{dy'}{dx'} = nax'^{n-1}$$

$$\therefore \mathrm{T}_{,} = y' \div nax'^{n-1} = \frac{x}{n}, \ \mathrm{N}_{,} = y' \ nax'^{n-1} = na^2x^{2n-1} \text{ or } \frac{ny'^2}{x'}.$$

3. To determine the subtangent at a given point in the logarithmic curve.

The equation of this curve related to rectangular coordinates is

$$y = a^x,$$

which shows that if the abscissas x be taken in arithmetical progression, the corresponding ordinates y will be in geometrical progression, so that the ordinates of this curve will represent the numbers, the logarithms of which are represented by the corresponding ab-

scissas, a being the assumed base of the system. Hence, calling the modulus of this base m, we have, by differentiating (13),

$$\frac{dy}{dx} = \frac{1}{m} y,$$

$$\therefore \mathrm{T}_{,} = y' \div \frac{y'}{m} = m.$$

Hence the remarkable property that the subtangent is constantly equal to the modulus of the system, whose base is a.

4. To determine the subtangent at a given point in the curve whose equation is

$$x^3 - 3axy + y^3 = 0.$$

Here

$$\left\{\frac{du}{dx}\right\} = 3x^2 - 3ay - 3ax\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0,$$

$$\therefore \frac{dy'}{dx'} = \frac{ay' - x'^2}{y'^2 - ax'},$$

$$\therefore \mathrm{T}_{,} = \frac{y'^3 - ax'y'}{ay' - x'^2}.$$

5. To draw a tangent to a rectangular hyperbola between the asymptotes, its equation being $xy = a$.

$$\mathrm{T}_{,} = x'.$$

6. To determine the subtangent at a given point in a curve whose equation is $y^m = ax^n$, which, because it includes the common parabola, is said to represent parabolas of all orders.

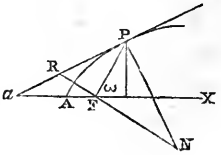
$$\mathrm{T}_{,} = \frac{m}{n} x'.$$

7. To determine the subtangent at a given point in a curve whose equation is $x^m y^n = a$, which, because it includes the common hyperbola, is said to belong to hyperbolas of all orders.

$$\mathrm{T}_{,} = \frac{n}{m} x'.$$

(80.) If the proposed curve be related to polar coordinates, then the expressions in last article must be changed into functions of these.

If the curve AP be related to polar coordinates $\mathrm{FP} = r$, $\mathrm{PFX} = \omega$, then if PR be a tangent at any point, P and PN the normal, and if RFN be perpendicular to the radius vector FP, the part FR will be the *polar subtangent*, and the part FN the *polar sub-*



normal. When the pole F and the point P are given, it is obvious that the determination of the subtangent FR will enable us to draw the tangent PR.

The formulas for transforming the equation of a curve from rectangular to polar coordinates, having the same origin, are (*Anal. Geom.*)

$$x = r \cos\ \omega,\ y = r \sin.\ \omega,\ x^2 + y^2 = r^2,$$

and the resulting equation of the curve will have the form

$$r = F\omega, \text{ or } F(r, \omega) = 0,$$

in which we shall consider ω as the independent variable. Now $RF = PF \cdot \tan.\ \angle P = r \tan.\ \angle P$, but by trigonometry,

$$\tan.\ \angle P = \frac{\tan.\ \omega - \tan.\ \alpha}{1 + \tan.\ \omega \tan.\ \alpha}$$

that is, since

$$\tan.\ \alpha = \frac{dy}{dx} \text{ and } \tan.\ \omega = \frac{y}{x}$$

$$\tan.\ \angle P = \frac{\frac{y}{x} - \frac{dy}{dx}}{1 + \frac{y}{x} \cdot \frac{dy}{dx}}$$

therefore r times this expression is the value of the polar subtangent. But the differential coefficient $\frac{dy}{dx}$, which implies that x is the principal variable, ought to become, when the variable is changed to ω, (66),

$$\frac{dy}{dx} = \frac{dy}{d\omega} \div \frac{dx}{d\omega} \therefore \tan.\ \angle P = \frac{y\frac{dx}{d\omega} - x\frac{dy}{d\omega}}{x\frac{dx}{d\omega} + y\frac{dy}{d\omega}}$$

Also, from the above formulas of transformation,

$$\frac{dx}{d\omega} = \frac{dr}{d\omega} \cos.\ \omega - r \sin.\ \omega,\ \frac{dy}{d\omega} = \frac{dr}{d\omega} \sin.\ \omega + r \cos.\ \omega$$

$$\therefore y\frac{dx}{d\omega} = r\frac{dr}{d\omega} \sin.\ \omega \cos.\ \omega - r^2 \sin.^2 \omega$$

$$x\frac{dy}{d\omega} = r\frac{dr}{d\omega} \sin.\ \omega \cos.\ \omega + r^2 \cos.^2 \omega$$

$$x \frac{dx}{d\omega} = r \frac{dr}{d\omega} \cos.^2 \omega - r^2 \sin. \omega \cos. \omega$$

$$y \frac{dy}{d\omega} = r \frac{dr}{d\omega} \sin.^2 \omega + r^2 \sin. \omega \cos. \omega$$

whence

$$y \frac{dx}{d\omega} - x \frac{dy}{d\omega} = -r^2, \; x \frac{dx}{d\omega} + y \frac{dy}{d\omega} = r \frac{dr}{d\omega}$$

consequently,

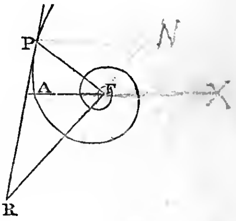
$$\tan. \angle \mathrm{P} = \frac{r}{\frac{dr}{d\omega}} \therefore \mathrm{RF} = \frac{r^2}{\frac{dr}{d\omega}} = \text{subtangent.}$$

Also

$$\mathrm{FN} = \frac{\mathrm{FP}^2}{\mathrm{FR}} = r^2 \div \frac{r^2}{\frac{dr}{d\omega}} = \frac{dr}{d\omega} = \text{subnormal.}$$

(81.) We shall apply these formulas to *spirals*, a class of curves always best represented by polar equations.

8. To determine the subtangent at any point in the *Logarithmic Spiral*, its equation being



$$r = a^\omega$$

$$\therefore \frac{dr}{d\omega} = \log. a \, a^\omega = \frac{a^\omega}{m}$$

$$\therefore r^2 \div \frac{dr}{d\omega} = mr = \mathrm{T}_,.$$

Hence, if a represent the base of the Napierian system, since the modulus will be 1, the subtangent will be equal to the radius vector, and therefore the angle P equal to 45°, because tan. $\angle$ P = 1.

Since, by the equation of this curve, log. $r = \omega$ log. a, it follows that, if a denote the base of any system, the various values of the angle or circular arc ω will denote the logarithms of the numbers represented by the corresponding values of r. Hence, the propriety of the name *logarithmic spiral*. In this curve

$$\tan. \angle \mathrm{P} = r \div \frac{dr}{d\omega} = a^\omega \div \frac{a^\omega}{m} = m\,;$$

hence the curve cuts all its radii vectores under the same angle.

9. To determine the subtangent at any point in the *Spiral of Archimedes*, its equation being

$$r = a\omega$$

$$\therefore \frac{dr}{d\omega} = a \therefore r^2 \div \frac{dr}{d\omega} = a\omega^2 = r\omega = \mathrm{T}_{,}$$

so that FR is equal to the length of the circular arc to radius r, comprehended between FR, FA; when, therefore, $\omega = 2\pi$, the subtangent equals the length of the whole circumference. The spiral of Archimedes belongs to the class of spirals represented by the general equation

$$r = a\omega^n.$$

When $n = -1$, we have $r\omega = a$, and the spiral represented is called the *hyperbolic spiral*, on account of the analogy between this equation and $xy = a$. It is also called the *reciprocal spiral.*

10. To determine the polar subtangent at any point in the hyperbolic spiral.

$$\mathrm{T}_{,} = a.$$

11. To determine the polar subtangent at any point in the spiral whose equation is $r = a\omega^{-\frac{1}{2}}$.

$$\mathrm{T}_{,} = 2a\omega^{\frac{1}{2}} = \frac{2a^2}{r}.$$

12. To determine the polar subtangent at any point in the *parabolic spiral*, its equation being $r = a\omega^{\frac{1}{2}}$.

$$\mathrm{T}_{,} = \frac{2r^3}{a^2}.$$

13. To determine the polar subtangent at any point in a spiral whose equation is

$$(r^2 - ar)\,\omega^2 - 1 = 0.$$

$$\mathrm{T}_{,} = \frac{\frac{1}{2}(a - 2r)\,r^2}{(r^2 - ar)^{\frac{3}{2}}}.$$

Rectilinear Asymptotes.

(82.) *A rectilinear asymptote* to a curve may be regarded as a tangent of which the point of contact is infinitely distant, so that the determination of the asymptote reduces to the determination of the

tangent on the hypothesis that either or both $y' = 0$, $x' = 0$, the portions of the axes between the origin and this tangent being, at the same time, one or both finite.

The equation of the tangent being

$$y - y' = \frac{dy'}{dx'}(x - x')$$

we have, by making successively $y = 0$, $x = 0$, the following expressions for the parts of the axes of x and y, between the tangent and the origin, viz.

$$x' - \frac{y'}{\frac{dy'}{dy'}} \text{ and } y' - x'\frac{dy'}{dx'} \quad . \quad . \quad . \quad . \quad (1).$$

If for $x = \infty$ both these are finite, they will determine two points, one on each axis, through which an asymptote passes. If for $x = \infty$ the first expression is finite and the second infinite, the first will determine a point on the axis of x, and the second will show that a line through this point and parallel to the axis of y is an asymptote. If, on the contrary, the second expression is finite, and the first infinite, the asymptote will pass through the point in the axis of y, determined by the finite value, and will be parallel to the axis of x.

When, however, asymptotes parallel to the axes exist, they may generally be detected by merely inspecting the equation, as it is only requisite to ascertain for what values of x, y becomes infinite, or for what values of y, x becomes infinite. Thus, in the equation $xy = a$, $x = 0$, renders $y = \infty$, and $y = 0$ renders $x = \infty$, therefore the two axes are asymptotes. Again, in the equation

$$a^2 y^3 - y^2 x^2 - bx^4 = 0, \text{ or } y^3 = \frac{bx^4}{a^2 - x^2}$$

it is plain that $x = \pm\, a$ renders $y = \infty$, we infer, therefore, that the curve represented by this equation has two asymptotes, each parallel to the axis of y, and at the distance a from it.

If both expressions are infinite, there will be no asymptote corresponding to $x = \infty$.

If both expressions are 0, the asymptote will pass through the origin, and its inclination θ, to the axis of x will be determined by $\frac{dy'}{dx'} =$ tan. θ.

If for $y = \infty$ one or both of the above expressions are finite, there will be an asymptote, and its position may be determined as in the foregoing cases.

EXAMPLES.

(83.) 1. Let the curve be the common hyperbola, of which the equation is

$$y = \frac{b}{a}\sqrt{x^2 - a^2}$$

$$\therefore \frac{dy}{dx} = \frac{bx}{a\sqrt{x^2 - a^2}}$$

hence the general expressions (82) are

$$x - \frac{x^2 - a^2}{x} = -\frac{a^2}{x}$$

and

$$-\frac{b}{a}\frac{a^2}{\sqrt{x^2 - a^2}} = -\frac{ba}{x\sqrt{1 - \frac{a^2}{x^2}}}$$

both of which are 0, when $x = \infty$; hence an asymptote passes through the origin.

Also

$$\frac{dy}{dx} = \pm\frac{b}{a}\cdot\frac{1}{\sqrt{1 - \frac{a^2}{x^2}}}$$

which becomes $\pm\frac{b}{a}$, when $x = \infty$, therefore, this being the tangent of the inclination of the asymptotes to the axis of x, they are both represented by the equation

$$y = \pm\frac{b}{a}x.$$

2. To prove that the hyperbola is the only curve of the second order that has asymptotes.

The general equation of a line of the second order, when referred to the principal diameter and tangent through the vertex as axes, is

$$y^3 = mx + nx^2,$$

$$\therefore x - \frac{y}{\frac{dy}{dx}} = x - \frac{2y^2}{m + 2nx} = \frac{mx + 2nx^2 - 2y^2}{m + 2nx} = - \frac{mx}{m + 2nx}$$

$$y - x\frac{dy}{dx} = y - \frac{mx + 2nx^2}{2y} = \frac{2y^2 - mx - 2nx^2}{2y} = \frac{mx}{2\sqrt{mx + nx^2}}.$$

Dividing numerator and denominator of each of these expressions by x, they reduce to

$$-\frac{m}{\frac{m}{x} + 2n} \text{ and } \frac{m}{2\sqrt{\frac{m}{x} + n}}$$

and these, when $x = \infty$, or indeed when $y = \infty$, become

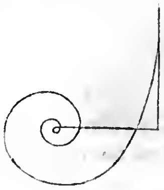
$$-\frac{m}{2n} \text{ and } \frac{m}{2\sqrt{n}}.$$

Hence the curve will have asymptotes, provided n be neither 0 nor negative, that is to say, provided the curve be neither a parabola nor an ellipse, but if it be either of these, there can exist no asymptote; therefore the hyperbola is the only line of the second order which has asymptotes.

(84.) When the curve is referred to polar coordinates, then, since the radius vector of the point of contact is infinite when the tangent becomes an asymptote, it follows that if for $r = \infty$ the subtangent is finite, this subtangent may be determined by (80) in terms of ω, and ω may be found from the equation of the curve, so that there will thus be determined a point in the asymptote and its direction, which is all that is necessary to fix its position. There will always be an asymptote if ω is finite, for $r = \infty$. If, for $r = \infty$, ω is also ∞, there exists no asymptote.

3. Let the curve be the hyperbolic spiral.

By ex. 10, art. 81, the subtangent at any point is constant, and equal to a, therefore there must be an asymptote; also by the equation of the curve $\omega = \frac{a}{r} = 0$, when $r = \infty$, therefore the asymptote is perpendicular to the fixed axis at the distance a from the pole.



Neither the logarithmic spiral, nor the spiral of Archimedes have an asymptote.

4. Let the spiral whose equation is

$$r = \frac{a\omega^2}{\omega^2 - 1} = \frac{a}{1 - \omega^{-2}}$$

be proposed, which admits of a rectilinear asymptote, because $\omega = 1$ renders $r = \infty$. The *direction*, therefore, of the asymptote is ascertained, and consequently the direction of the infinite radius vector, since they must be parallel. It remains, therefore, to determine the subtangent, or distance of the asymptote from the pole

$$\frac{dr}{d\omega} = -\frac{2a\omega^{-3}}{(1 - \omega^{-2})^2} = -\frac{2r^2}{a\omega^3}$$

$$\therefore r^2 \div \frac{dr}{d\omega} = r^2 \div \frac{2r^2}{a\omega^3} = \frac{a\omega^3}{2} = \frac{a}{2} = \mathrm{T},$$

because $\omega = 1$ when $r = \infty$.

(85.) Although we do not propose to treat fully in this place of *curvilinear* asymptotes, yet we may remark in passing, that if r should be finite although ω be infinite, it will prove that the spiral must be continued for an infinite number of revolutions round its pole, before it can meet the circumference of a circle whose radius is this finite value. In such a case, therefore, the spiral has a *circular asymptote*. If, moreover, the value of r for $\omega = \infty$ be greater than the value of r for every other value of ω, the spiral will be included within its circular asymptote, but, otherwise, it will be without this circle.

5. Thus in the spiral whose equation is

$$(r^2 - ar)\,\omega^2 - 1 = 0 \text{ or } \omega = \frac{1}{\sqrt{r^2 - ar}}$$

ω is infinite when $r = a$, and for all less values of r, ω is imaginary; hence the spiral can never approach so near to the pole as $r = a$, till after an infinite number of revolutions, so that the circumference whose radius is a is *within* the spiral and is asymptotic.

If, on the contrary, the equation had been

$$\omega = \frac{1}{\sqrt{ar - r^2}}$$

then also $r = a$ gives $\omega = \infty$, but for all other real values of ω, r is less than a, so that this spiral is enclosed by its asymptotic circle, the radius of which is a.

6. To determine the rectilinear asymptote to the logarithmic curve.

The axis of x.

7. To determine the equation of the asymptote to the curve whose equation is

$$y^3 = ax^2 + x^3.$$

The equation is $y = x + \frac{1}{3} a$.

8. To determine the rectilinear asymptote to the spiral whose equation is

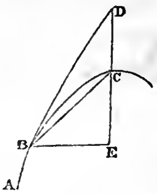
$$r = a\omega^{-\frac{1}{2}}.$$

The fixed axis is the asymptote.

9. To determine whether the spiral shown to have a rectilinear asymptote in ex. 4 has also a circular asymptote.

The circle whose centre is the pole and radius $= a$ is an asymptote.

(86.) Before terminating the present chapter, it will be necessary to exhibit the expression for the differential of the arc of any plane curve, as we shall have occasion to employ this expression in the next chapter.



Let us call the arc AB of any plane curve s, and the coordinates of B, x, y; let also BD be a tangent at B, and BC any chord, then if BE, ED are parallel to the rectangular axes, BC will be the increment of the arc s corresponding to BE $= h$, the increment of the abscissa x.

Now, putting tan. DBE $= \alpha$, we have

$$\text{ED} = h\alpha \therefore \text{BD} = \sqrt{h^2 + h^2\alpha^2}$$

and

$$\frac{\text{BD} + \text{DC}}{\text{BC}} = \frac{\sqrt{h^2(1+\alpha^2)} + h\alpha - \text{CE}}{\sqrt{h^2 + \text{CE}^2}}$$

This ratio continually approaches to $\frac{\text{CE}}{\text{CE}}$ or to unity as h diminishes and this it actually becomes when $h = 0$. Consequently, since the arc BC is always, when of any definite length, longer than the chord

BC and shorter than BD + BC,* it follows that when $h = 0$ that the ratio of the arc to either of these must be unity; therefore

$$\text{in the limit } \frac{\text{arc BC}}{\text{chord BC}} = 1 \ \therefore \ \frac{\text{arc BC}}{h} \div \frac{\text{chord BC}}{h} = 1,$$

but

$$\frac{\text{chord BC}}{h} = \frac{\sqrt{h^2 + \text{CE}^2}}{h} = \sqrt{1 + \frac{\text{CE}^2}{h^2}},$$

and CE is the increment of the ordinate y corresponding to the increment h of the abscissa x; hence, when $h = 0$, the ratio becomes

$$\frac{ds}{dx} \div \sqrt{1 + \frac{dy^2}{dx^2}} = 1$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + \frac{dy^2}{dx^2}}.$$

If any other independent variable be taken instead of x, then, denoting the several differential coefficients relatively to this new variable by (dx), (dy), (ds) we have (66)

$$(ds) = \sqrt{(dx)^2 + (dy)^2}.$$

At the point where $\frac{dy}{dx} = 0$, $\frac{ds}{dx} = 1$, or $(ds) = (dx)$.

CHAPTER II.

ON OSCULATION, AND THE RADIUS OF CURVATURE OF PLANE CURVES.

(87.) Let

$$y = fx, \ \text{Y} = \text{F}x,$$

be the equations of two plane curves, in the former of which we shall suppose the constants a, b, c, &c. to be known, and therefore the curve itself to be determinate; while in the latter we shall consider the constants A, B, C, &c. to be unknown, or arbitrary, and there-

* See Young's *Elements of Plane Geometry*.

fore the *species* only of the curve given. The constants which enter into the equation of a curve, are usually called the *parameters*.

If, now, x take the increment h, and the corresponding ordinates y', Y' be developed, we shall have, by Taylor's theorem,

$$y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{1\cdot 2} + \frac{d^3y}{dx^3}\frac{h^3}{1\cdot 2\cdot 3} + \&c. \ldots \ldots (1),$$

$$Y' = Y + \frac{dY}{dx}h + \frac{d^2Y}{dx^2}\frac{h^2}{1\cdot 2} + \frac{d^3Y}{dx^3}\frac{h^3}{1\cdot 2\cdot 3} + \&c. \ldots \ldots (2).$$

Now, the parameters which enter (2) being arbitrary, they may be determined so as to fulfil as many of the conditions

$$y = Y, \frac{dy}{dx} = \frac{dY}{dx}, \frac{d^2y}{dx^2} = \frac{d^2Y}{dx^2}, \&c. \ldots . (3),$$

as there are parameters, but obviously not more conditions.

We shall thus have the values of A, B, C, &c. in terms of x, and of the fixed parameters a, b, c, &c.; which values, substituted in (2), will cause so many of the leading terms in both series to become *identical*, whatever be the value of x. Other corresponding terms of the two series may, indeed, be rendered also identical, but this can take place only accidentally, not necessarily. Hence, whatever particular value we now give to x, the resulting values of the corresponding coefficients will necessarily agree to the extent mentioned, that is, as far as the n first terms, if there are n constants originally in (2); and this is true, even if such particular value of x render any of the coefficients infinite, inasmuch as they are *always identical* as far as these terms, but no further.

We know, however, that in those cases where any of the coefficients become infinite, (1) and (2) will fail to represent the true developments of the ordinates y', Y' at the proposed points. Nevertheless, as the two series have been rendered identical, as far as n terms, should they both fail within this extent, the terms which supply these in the true development, must necessarily be identical. (*See note C at the end.*)

Now the greater the number of leading terms in the two developments, which are identical, the nearer will the developments themselves approach to identity, provided, at least, h may be taken as small as we please; for if $n - 1$ terms in each are identical, we may represent the difference of the two developments by

$$A_n h^{\alpha} + S - (A'_n h^{\alpha'} + S') \ldots\ldots (4),$$

where S, S′ represent the sums of the remaining terms in each series after the nth. Hence, h^{α} being the highest power of h which enters this expression, for the difference it follows from (47), that a value may be given to h small enough to cause the term $A_n h^{\alpha}$ to become greater than all the other terms in (4), and consequently, for this small value,

$$A_n h^{\alpha} - A'_n h^{\alpha'} > S - S',$$

and, therefore, the whole difference (4) is greater than twice S — S′, but when the nth term is the same in both developments, as well as the preceding terms, then the difference (4) is reduced simply to S — S′, which we have just seen to be less than (4). Consequently the developments approach nearer to identity, for all values of h between some certain finite value h' and 0 as the number of identical leading terms become greater.

When the first of the conditions (3) exist, the curves have a common point; when the second also exists they have a common tangent at that point, and are consequently in contact there, and the contact will be the more intimate, or the curves will be the closer in the vicinity of the point, as the number of following conditions become greater; so that of all curves of a given species, that will touch any fixed curve at a proposed point with the closest contact whose parameters are *all* determined agreeably to the conditions (3). No other curve of the same species can, from what is proved above, approach so nearly to coincidence with the proposed, in the immediate vicinity of the point of contact, as this; so that no other of that species can pass between this and the proposed. A curve, thus determined, is said to be, in reference to the proposed curve, its *osculating curve* of the given species.

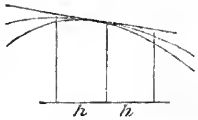
(88.) It appears, from what has now been said, that there may be different *orders* of contact at any proposed point. The two first of the conditions (3) must exist for there to be contact at all; therefore, when only these exist, the contact is called *simple contact*, or *contact of the first order*; if the next condition also exist, the contact is of the *second order*, and so on; and it is obvious, that of any given species, the osculating curve will have the highest order of contact.

at any proposed point, in a given curve. If the curve, given in species, has n parameters, the highest order of contact will be the $n - 1$th, unless, indeed, the same values of these parameters that fulfil the n conditions (3), should happen also to fulfil the $n + 1$th, the $n + 2$th, &c.; but this, as observed before, can take place only accidentally, and cannot be predicted of any proposed point, although we see it is possible for such points to exist.

(89.) At those points in the proposed curve, for which Taylor's development does not fail, contact of an even order is both contact and intersection, and contact of an odd order is without intersection; before proving this, however, we may hint to the student that contact is not opposed to intersection, for two curves are said to be in contact at a point, when they have a common tangent at that point; and yet, as we are about to show, one of these curves may pass between the tangent and the other, and so intersect where they are admitted to be in contact. To prove the proposition, let us take the difference (4), which, when Taylor's theorem holds, is

$$(A_n - A'_n)\, h^{\alpha} + S - S' \ldots\ldots (5,)$$

A_n A'_n being here the $n - 1$th differential coefficients. If these are *odd*, the contact is of an even order, also α being odd, h^{α} will have contrary signs for $h = + h'$ and $h = - h'$, and therefore, since for these small values of h, the sign of the whole expression (5) is the same as that of the first term, the differences of the ordinates corresponding to $x + h$, and to $x - h$, will be the one positive and the other negative, so that the two curves must necessarily cross at the point whose abscissa is x.



But if α is even, the contact is of an odd order, and the difference (5) between the ordinates of the two curves corresponding to the same abscissa, $x + h$, will, for a small value of h, have the same sign, whether h be positive or negative; so that, in this case, the curves do not cross each other at the point of contact.

(90.) The student must not fail to bear in remembrance, that the proposition just established, comprehends only those points of the proposed curve, at which none of the differential coefficients become infinite from the first to that immediately beyond the coefficient which fixes the order of the contact. For it is only upon the supposition

that the true development, within the limits, proceeds according to the ascending integral and positive powers of h, that the foregoing conclusions respecting the signs of the difference (5) can be fairly drawn. (*See note* C.)

(91.) From the principles of osculation now established, it is evident that any plane curve being given, and any point in it chosen, we may always find what particular curve, of any proposed species, shall touch at that point with the closest contact, or which shall most nearly coincide with the given curve in the immediate vicinity of the proposed point. Thus an *ellipse* or a *parabola* being given, and a point in it proposed, we may determine the *circle* that shall approach more nearly to coincidence with that ellipse or parabola in the vicinity of the proposed point, than any other circle, and which will therefore better represent the curvature of the given curve at the proposed point than any other. On account of its simplicity and uniformity, the circle is the curve employed to estimate, in this way, the curvature of other curves at proposed points; that is, the curvature is estimated by the curvature of the *osculating circle*, or rather as the curvature of a circle increases as the radius diminishes, and vice versa, it is usual to adopt, as a fit representation of the curvature, the reciprocal of the radius.

The osculating circle is called the *circle of curvature*, and its radius the *radius of curvature*, and, from what has been said above, it follows that the determination of the curvature at any point in a proposed curve, reduces itself to the determination of this radius: to this, therefore, we shall now proceed.

Radius of Curvature.

PROBLEM I.

(92.) To determine the radius of curvature at any proposed point of a given curve.

The general equation of a circle being

$$(x - \alpha)^2 + (y - \beta)^2 = r^2,$$

it becomes determined as soon as we fix the values of the parameters α, β, r, and these may be determined, so as to fulfil any three independent conditions, but not more. In the case before us, the conditions to be fulfilled are those of (3) art. (87), that is to say, putting

p', p'', &c. for the successive differential coefficients derived from $\mathrm{Y} = \mathrm{F}x$, the equation of the given curve, the conditions to be fulfilled are

$$y = \mathrm{Y}, \frac{dy}{dx} = p', \frac{d^2y}{dx^2} = p'', .$$

in order that the resulting values of α, β, r, may belong to the equation of the osculating circle. Now

$$\frac{dy}{dx} = -\frac{x-\alpha}{y-\beta}, \frac{dy^2}{dx^2} = -\frac{1}{y-\beta} - \frac{(x-\alpha)^2}{(y-\beta)^3} = -\frac{r^2}{(y-\beta)^3},$$

hence the three equations for determining α, β and r, are

$$(x-\alpha)^2 + (y-\beta)^2 = r^2 \ldots . (1),$$

$$(x-\alpha) + p'(y-\beta) = 0 \ldots . (2),$$

$$p''(y-\beta)^3 = -r^2 \ldots . (3)$$

From the second equation

$$(x-\alpha)^2 = p'^2(y-\beta)^2.$$

Substituting this in the first,

$$(p'^2+1)(y-\beta)^2 = r^2.$$

Adding this last to the third, there results

$$y-\beta = -\frac{p'^2+1}{p''},$$

which, substituted in (2), gives

$$x-\alpha = \frac{p'(p'^2+1)}{p''} \therefore r^2 = -\frac{(p'^2+1)^3}{p''^2}.$$

Consequently,

$$\alpha = x - \frac{p'(p'^2+1)}{p''}, \beta = y + \frac{p'^2+1}{p''}$$

$$r = -\frac{(p'^2+1)^{\frac{3}{2}}}{p''} = -\frac{\frac{ds^3}{dx^3}}{p''}.$$

These equations completely determine the osculating circle, whenever the co-ordinates x, y of the proposed point are given.

Should this point be such as to render $p' = 0$, then the expression, for the radius of curvature at that point, becomes

$$r = -\frac{1}{p''} = -\frac{1}{\frac{d^2y}{dx^2}}.$$

But when $p' = 0$, the tangent at the proposed point must be parallel to the axis of x (78), or, which is the same thing, the axis of y must coincide with the normal; hence, under this arrangement of the axes, $x = 0$ at the proposed point, and therefore

$$r = -\frac{1}{\left[\frac{d^2y}{dx^2}\right]}.$$

Should $p'' = 0$ at the proposed point, r will be infinite, whether $p' = 0$ or not, so that the osculating circle then becomes a straight line; as, therefore, this straight line has contact of the second order,

the parts of the curve in the vicinity of the point will lie on contrary sides of it, as in the annexed diagram (89), that is, supposing p''' is neither 0 nor ∞. If $p''' = 0$, and the next following differential coefficient neither 0 nor ∞, the contact will be of an order which is unaccompanied by intersection.

A point at which the tangent intersects the curve, or at which the curve changes from convex to concave, is called *a point of inflexion*, or, *a point of contrary flexure*. The analytical indications of such points will be more fully inquired into, when we come to speak of the *singular points* of curves.

(93.) By referring to equation (2) above, which has place even when the contact is but of the first order, we learn that the centre (α, β) of every touching circle, is always on the normal at the point of contact; for that equation is the same as

$$(\alpha - x) = -\frac{1}{\frac{dy}{dx}}(\beta - y).$$

We shall now apply the general expression, for the radius of curvature, to a few particular cases.

EXAMPLES.

(94.) 1. To determine the radius of curvature, at any point in a parabola.

Differentiating the equation of the curve,

$$y^2 = 4mx,$$

we have,

$$2yp' = 4m \therefore p' = \frac{2m}{y}$$

$$2yp'' + 2p'^2 = 0 \therefore p'' = \frac{p'^2}{y} = -\frac{4m^2}{y^3}$$

$$\therefore r = -\frac{(p'^2+1)^{\frac{3}{2}}}{p''} = \left(\frac{m+x}{x}\right)^{\frac{3}{2}} \cdot \frac{y^3}{4m^2} = (m^2+mx)^{\frac{3}{2}}\frac{2}{m^2}$$

$$= \frac{(\text{normal})^3}{4m^2} \text{ (See } \mathit{Anal.\ Geom.}\text{)}$$

As the expression for the normal diminishes with x, the vertex is the point of greatest curvature, r being there equal to $2m$, or to half the parameter.

2. To determine the radius of curvature at any point in an ellipse.

By differentiating the equation

$$a^2y^2 + b^2x^2 = a^2b^2,$$

we have

$$a^2yp' + b^2x = 0 \therefore p' = -\frac{b^2x}{a^2y}$$

$$a^2yp'' + a^2p'^2 + b^2 = 0 \therefore p'' = -\frac{b^2 + a^2p'^2}{a^2y} = -\frac{b^4}{a^2y^3}$$

$$\therefore r = -\frac{(p'^2+1)^{\frac{3}{2}}}{p''} = \frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^6y^3} \cdot \frac{ay^3}{b^4} = \frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4} \quad \ldots (1).$$

From this expression, others occasionally useful may be readily derived. Thus, since (*Anal. Geom.*) the square of the normal, N, is $\frac{b^4}{a^4}x^2 + y^2$, therefore,

$$a^4\mathrm{N}^2 = b^4x^2 + a^4y^2 \therefore r = \frac{a^6\mathrm{N}^3}{a^4b^4} = \frac{a^2}{b^4}\mathrm{N}^3 \quad \ldots\ldots (2).$$

Again, since (*Anal. Geom.*),

$$a\mathrm{N} = bb' \therefore r = \frac{b'^3}{ab} \quad \ldots\ldots (3).$$

At the vertex $r = \frac{b^2}{a} =$ semiparameter (*Anal. Geom.*)

From equations (2) and (3) it follows that, in the ellipse, the radius of curvature varies as the cube of the normal, or as the cube of the diameter parallel to the tangent through the proposed point.

It is often desirable to obtain r as a function of λ, the angle included between the normal and the transverse axis. For this purpose we have since

$$x^2 = a^2 (1 - \frac{y^2}{b^2}) \text{ and } y^2 = \text{N}^2 \sin.^2\lambda$$

$$\text{N}^2 = \frac{b^4}{a^2} (1 - \frac{\text{N}^2 \sin.^2\lambda}{b^2}) + \text{N}^2 \sin.^2\lambda,$$

$$\therefore \text{N}^2 \{1 - (1 - \frac{b^2}{a^2}) \sin.^2\lambda\} = \frac{b^4}{a^2}$$

but (*Anal. Geom.*)

$$1 - \frac{b^2}{a^2} = e^2$$

$$\therefore \text{N} = \frac{b^2}{a} \cdot \frac{1}{(1 - e^2 \sin.^2\lambda)^{\frac{1}{2}}}$$

$$\therefore r = \frac{a^2}{b^4} \text{N}^3 = \frac{b^2}{a (1 - e^2 \sin.^2\lambda)^{\frac{3}{2}}} = \frac{a (1 - e^2)}{(1 - e^2 \sin.^2\lambda)^{\frac{3}{2}}}$$

(95.) Since, in the ellipse, the principal transverse is the longest diameter, and its conjugate the shortest (*Anal. Geom.*), it follows from (3), that the curvature $\frac{1}{r}$ is greatest at the vertex of the transverse, and least at the vertex of the conjugate axis. At the former point $r = \frac{b^2}{a}$, and at the latter $r = \frac{a^2}{b}$.

The present is a very important problem, being intimately connected with inquiries relative to the figure of the earth.

By means of the last expression for r, the ratio of the polar and equatorial diameters of the earth, may be readily deduced, when we know the lengths of a degree of the meridian in two known latitudes, L, l, for these lengths may, without error, be considered to coincide with the osculating circles through their middle points; and since

similar arcs of circles are as their radii, we have, by putting M, m for the measured degrees, and R, r for the corresponding radii,

$$R : r :: M : m,$$

but

$$R = \frac{a(1-e^2)}{(1-e^2 \sin.^2 L)^{\frac{3}{2}}} \text{ and } r = \frac{a(1-e^2)}{(1-e^2 \sin.^2 l)^{\frac{3}{2}}},$$

therefore, since $mR = Mr$, we have

$$\frac{m}{(1-e^2 \sin.^2 L)^{\frac{3}{2}}} = \frac{M}{(1-e^2 \sin.^2 l)^{\frac{3}{2}}},$$

or

$$m^{\frac{2}{3}}(1-e^2 \sin.^2 l) = M^{\frac{2}{3}}(1-e^2 \sin.^2 L),$$

$$\therefore e^2 = 1 - \frac{b^2}{a^2} = \frac{M^{\frac{2}{3}} - m^{\frac{2}{3}}}{M^{\frac{2}{3}} \sin.^2 L - m^{\frac{2}{3}} \sin.^2 l}$$

$$\therefore \frac{a}{b} = \sqrt{\left\{ \frac{M^{\frac{2}{3}} \sin.^2 L - m^{\frac{2}{3}} \sin.^2 l}{m^{\frac{2}{3}} \cos.^2 l - M^{\frac{2}{3}} \cos.^2 L} \right\}}$$

$$= \sqrt{\left\{ \frac{\sin.^2 L - \left(\frac{m^{\frac{2}{3}}}{M}\right) \sin.^2 l}{\left(\frac{m}{M}\right)^{\frac{2}{3}} \cos.^2 l - \cos.^2 L} \right\}}.$$

If $l = 0$, that is, if the degree m is measured at the equator, then,

$$\frac{a}{b} = \sqrt{\left\{ \frac{\sin. L}{\left(\frac{m}{M}\right)^{\frac{2}{3}} - \cos.^2 L} \right\}}.$$

3. To determine the radius of curvature at any point in the logarithmic curve, its equation being $y = a^x$,

$$r = \frac{(m^4 + y^2)^{\frac{3}{2}}}{my}, \ m \text{ being the modulus.}$$

4. To determine the radius of curvature at any point in the cubical parabola, its equation being $y^3 = ax$.

$$r = \frac{(9y^4 + a^2)^{\frac{3}{2}}}{6a^2 y}.$$

PROBLEM II.

(96.) To determine these points in a given curve, at which the osculating circle shall have contact of the third order.

It is here required to find for what points of a given curve the values of α, β, r, determined by the three first conditions (3), art. (87) satisfy also the fourth condition.

The differential coefficient p''' as derived from equation (3), p. 131, is

$$p''' = -\frac{3p''\,p'}{y-\beta},$$

and this must agree with the p''' derived from the equation of the proposed curve, at those points where the contact is of the third order; that is, the abscissas of these points will all be given by the roots of the equation

$$(y-\beta)\,p''' + 3p''\,p' = 0,$$

and it may be easily shown, that the points which satisfy this equation are those of greatest and least curvature, for since

$$r = -\frac{(p'^2+1)^{\frac{3}{2}}}{p''}$$

$$\therefore \frac{dr}{dx} = \frac{-3\,(p'^2+1)^{\frac{1}{2}}\,p'\,p''^2 + (p'^2+1)^{\frac{3}{2}}\,p'''}{p''^2}$$

and when r is a maximum or a minimum this expression is equal to 0 (49); hence

$$-3p'\,p''^2 + (p'^2+1)\,p''' = 0,$$

or, dividing by p'' and recalling the value of $y-\beta$ deduced in (92), we have, finally,

$$(y-\beta)\,p''' + 3p''\,p' = 0,$$

which being the same equation as that deduced above, it follows that the points of maximum and minimum curvature are the same as those at which the contact is of the third order.

(97.) In the preceding investigations we have always considered x to be the independent variable, because the expression for the radius of curvature has been obtained conformably to this hypothesis. But if any other quantity is taken for the independent variable, the

foregoing expression for r will not apply; therefore, in order to give the greatest generality possible to the formula for the radius of curvature, we shall now suppose any arbitrary quantity whatever to be the independent variable, x and y being functions of it. Hence, instead of p' and p'', we shall have (66)

$$\frac{(dy)}{(dx)} \text{ and } \frac{(d^2y)\,(dx) - (d^2x)\,(dy)}{(dx^3)}$$

the parentheses intending to intimate that the independent variable, according to which the differentials of the functions x, y are taken, is arbitrary, and the differential of which when chosen is with its proper powers to be introduced as denominators of the above differentials. Making, therefore, these substitutions in the expression for r, it becomes

$$r = -\frac{\left((dy)^2 + (dx)^2 \right)^{\frac{3}{2}}}{(d^2y)\,(dx) - (d^2x)\,(dy)}$$

or, since (86)

$$(ds) = \sqrt{(dy)^2 + (dx)^2},$$

whatever be the independent variable,

$$r = -\frac{(ds)^3}{(d^2y)\,(dx) - (d^2x)\,(dy)} \quad \ldots\ldots (1).$$

(98.) This expression is of the utmost generality, and will furnish a correct formula for every hypothesis respecting the independent variable. Thus, if x be chosen for the independent variable, then $(dx) = 1$ and $(d^2x) = 0$, and the formula in that case is

$$r - \frac{\frac{ds^3}{dx^3}}{\frac{d^2y}{dx^2}} \quad \ldots\ldots (2).$$

being the same as that at first given as it ought to be. If y be the independent variable, then $(dy) = 1$ and $(d^2y) = 0$, so that upon this hypothesis the formula is

$$r = +\frac{\frac{ds^3}{dy^3}}{\frac{d^2x}{dy^2}} \quad \ldots\ldots (3).$$

If s be the independent variable, then $(ds) = 1$;

$$\therefore (d^2s) = d\sqrt{(dx)^2+(dy)^2} = 0 \therefore (d^2y) = -\frac{(dx)}{(dy)}(d^2x) \ldots (4).$$

substituting this in the denominator of (1) we have

$$r = -\frac{(dy)}{(d^2x)} \div \overline{(dx)^2 + (dy)^2} = -\frac{(dy)}{(d^2x)} = -\frac{\frac{dy}{ds}}{\frac{d^2x}{ds^2}} \quad \ldots . \ (5).$$

By squaring (1) on this last hypothesis we have

$$r^2 = \frac{1}{\overline{(d^2y)\,(dx) - (d^2x)\,(dy)}\,|^2}$$

but, since from (4)

$$(d^2y)\,(dy) + (d^2x)\,(dx) = 0,$$

it may be added to the denominator of this expression for r^2 without affecting its value, so that

$$r^2 = \frac{1}{\overline{(d^2y)\,(dx) - (d^2x)\,(dy)}\,|^2 + \overline{(d^2y)\,(dy) + (d^2x)\,(dx)}\,|^2}$$

$$= \frac{1}{\overline{(dy)^2 + (dx)^2} \times \overline{(d^2y)^2 + (d^2x)^2}}$$

$$= \frac{1}{(d^2y)^2 + (d^2x)^2} \therefore r = \frac{1}{\sqrt{(\frac{d^2y}{ds^2})^2 + (\frac{d^2x}{ds^2})^2}} \quad \ldots . \ (6).$$

(99.) We shall now proceed to determine a suitable formula for *polar curves.*

If the circle whose equation is (1) p. 131, be transformed from rectangular to polar coordinates, the pole being at the origin of the primitive axes, and the axis of x being the fixed line from which the variable angle ω of the radius vector γ is measured, we shall have (*Anal. Geom.*)

$$(\gamma \cos. \omega - \alpha)^2 + (\gamma \sin. \omega - \beta^2) = r^2 \quad \ldots . \ (1).$$

If, therefore, we differentiate on the supposition that ω is the independent and γ the dependent variable, and denote the first and second differential coefficients by $p_{,}$ and $p_{,,}$, we shall have

$(\gamma \cos.\omega - a)(p_{,}\cos.\omega - \gamma \sin.\omega) + (\gamma \sin.\omega - \beta)(p_{,}\sin.\omega + \gamma\cos.\omega) = 0 \ldots (2)$
$(p_{,}\cos.\omega - \gamma\sin.\omega)^2 + (\gamma\cos.\omega - a)(p_{,,}\cos.\omega - 2p_{,}\sin.\omega - \gamma\cos.\omega) +$
$(p_{,}\sin.\omega + \gamma\cos.\omega)^2 + (\gamma\sin.\omega - \beta)(p_{,,}\sin.\omega + 2p_{,}\cos.\omega - \gamma\sin.\omega) = 0 \ldots (3)$

If from the two latter equations we determine the values of γ sin. $\omega - \beta$ and γ cos. $\omega - \alpha$, and substitute them in (1), we shall obtain the following expression for r in functions of γ and its differential coefficients, viz.

$$r = \frac{(\gamma^2 + p_{,}^2)^{\frac{3}{2}}}{\gamma^2 + 2p_{,}^2 - \gamma p_{,,}} \quad . \quad . \quad . \quad . \quad (4)$$

But we shall arrive at this expression more readily by first deducing from the equations

$$y = \gamma \sin.\omega, \; x = \gamma \cos.\omega$$

the differential coefficients

$$\frac{dy}{d\omega} = \gamma \cos.\omega + p_{,}\sin.\omega = (dy)$$

$$\frac{dx}{d\omega} = -\gamma\sin.\omega + p_{,}\cos.\omega = (dx)$$

$$\frac{d^2y}{d\omega^2} = -\gamma\sin.\omega + 2p_{,}\cos.\omega + p_{,,}\sin.\omega = (d^2y)$$

$$\frac{d^2x}{d\omega^2} = -\gamma\cos.\omega - 2p_{,}\sin.\omega + p_{,,}\cos.\omega = (d^2x)$$

and then substituting them in the general formula (1).

Since (80) the expression for the normal PN is

$$\mathrm{N} = \gamma^2 + p_{,}^2)^{\frac{1}{2}},$$

we may put the above expression for r under the form

$$r = \frac{\mathrm{N}^3}{r^2 + 2p_{,}^2 - \gamma p_{,,}} \quad . \quad . \quad . \quad . \quad (5).$$

5. To determine the radius of curvature at any point in the logarithmic spiral

$$\gamma = a^{\omega}$$

$$\frac{d\gamma}{d\omega} = \frac{a^{\omega}}{m} = \frac{\gamma}{m} = p_{,}$$

$$\frac{d^2\gamma}{d\omega} = \frac{\gamma}{m^2} = p_{,,}.$$

Hence

$$r = \frac{(\gamma^2 + p_{,}^2)^{\frac{3}{2}}}{\gamma^2 + 2p_{,}^2 - \gamma p_{,,}} = (\gamma^2 + p_{,}^2)^{\frac{1}{2}} = \gamma \sqrt{1 + \frac{1}{m^2}} =$$

$$\gamma \sqrt{1 + \frac{1}{\tan.^2 \text{P}}} \ (art.\ 80) = \gamma \text{ cosec. P}.$$

It appears, therefore, that the radius of curvature is always equal to the normal.

6. To determine the radius of curvature at any point in the curve whose equation is

$$\gamma = 2 \cos. \omega \pm 1$$

$$\therefore r = \frac{(5 \pm 4 \cos. \omega)^{\frac{3}{2}}}{9 \pm 6 \cos. \omega}.$$

CHAPTER III.

ON INVOLUTES, EVOLUTES, AND CONSECUTIVE CURVES.

(100.) If osculating circles be applied to *every* point in a curve, the locus of their centres is called the *evolute* of the proposed curve, this latter being called the *involute.*

The equation of the evolute may be determined by combining the equation of the proposed curve with the equations (2), (3) p. **131**, containing the *variable* coordinates α, β of the centre. As these three equations must exist simultaneously for every point of contact (x, y), the two quantities x, y may be eliminated, and therefore, a resulting equation obtained containing only α and β, which equation therefore will express the general relation between α and β for every point (x, y); in other words, it will represent the locus of the centres of the osculating circles.

Or, representing the equation of the proposed curve by $y = \text{F}x$, we shall have to eliminate x and y from the equations (p. **131**)

$$\alpha = x - \frac{p'(p'^2+1)}{p''},\ \beta = y + \frac{p'^2+1}{p''}$$

$$y = \mathrm{F}x,$$

when the resulting equation in α, β will be that of the evolute.

EXAMPLES.

(101.) To determine the evolute of the common parabola

$$y^2 = 4mx \mathrel{.\cdot.} p' = \frac{2m}{y} \mathrel{.\cdot.} p'' = -\frac{4m^2}{y^3}$$

$$\therefore 1 + p'^2 = \frac{y^2+4m^2}{y^2} = 1 + \frac{m}{x},\ \frac{p'}{p''} = -\frac{y^2}{2m}$$

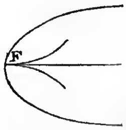
$$\therefore \alpha = x + \frac{y^2}{2m} + 2m = 3x + 2m \mathrel{.\cdot.} x = \frac{\alpha - 2m}{3}$$

$$\beta = y - \frac{y^3}{4m^2} - y = \frac{-y^3}{m\frac{y^2}{x}} = \frac{-2x^{\frac{3}{2}}}{m^{\frac{1}{2}}} \mathrel{.\cdot.} x = \left(\frac{m\beta^2}{4}\right)^{\frac{1}{3}}$$

$$\therefore \beta^2 = \frac{4}{27}(\alpha - 2m)^3,$$

which is the equation of the evolute. If the origin be removed to that point in the axis of x whose abscissa is $2m$, then the equation becomes

$$\beta^2 = \frac{4}{27}\alpha^3,$$



The locus of which is called the *semicubical parabola.* It passes through the origin because $\beta = 0$ when $\alpha = 0$; therefore the focus of the proposed involute is in the middle, between its vertex and the vertex of the evolute. (*Anal. Geom.* art. 100.) The curve consists of two branches symmetrically situated with respect to the axis of x or of α, and lies entirely to the right of the origin, for every positive value of α gives two equal and opposite values of β, and for negative values of α, β is impossible. It is easy to see, therefore, that the form of the curve is that represented in the margin.

2. To determine the evolute of the ellipse.

By example 2, page 133, we have

$$p' = -\frac{b^2x}{a^2y},\ p'' = -\frac{b^4}{a^2y^3}$$

$$\therefore 1 + p'^2 = \frac{a^4y^2 + b^4x^2}{a^4y^2},\ \frac{p'}{p''} = \frac{xy^2}{b^2}$$

$$\therefore \alpha = x - \frac{x(a^4y^2 + b^4x^2)}{a^4b^2},\ \beta = y - \frac{y(a^4y^2 + b^4x^2)}{a^2b^4}.$$

Now, since, by the equation of the curve,

$$a^2y^2 = a^2b^2 - b^2x^2 \text{ or } b^2x^2 = a^2b^2 - a^2y^2$$

$$\therefore a^4y^2 + b^4x^2 = b^2(a^4 - c^2x^2) \text{ or } = a^2(b^4 + c^2y^2),$$

c^2 being put for $a^2 - b^2$. Hence, by substitution,

$$\alpha = \frac{c^2x^3}{a^4},\ \beta = -\frac{c^2y^3}{b^4}$$

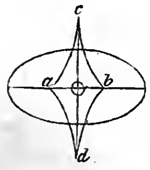
$$\therefore x^2 = \left(\frac{\alpha^2a^3}{c^4}\right)^{\frac{1}{3}},\ y^2 = \left(\frac{\beta^2b^3}{c^4}\right)^{\frac{1}{3}}.$$

Substituting these values in the equation of the involute, we have

$$a^2\left(\frac{\beta^2b^3}{c^4}\right)^{\frac{1}{3}} + b^2\left(\frac{\alpha^2a^8}{c^4}\right)^{\frac{1}{3}} = a^2b^2,$$

or, finally, dividing all the terms by $\frac{a^2b^2}{c^{\frac{4}{3}}}$, we obtain for the evolute the equation

$$(b\beta)^{\frac{2}{3}} + (a\alpha)^{\frac{2}{3}} = c^{\frac{4}{3}} = (a^2 - b^2)^{\frac{4}{3}}.$$



If $\alpha = 0$, then $\beta = \pm\frac{c^2}{b}$, so that the curve meets the axis of y in two points, c, d, equidistant from the origin O. If $\beta = 0$, then $\alpha = \pm\frac{c^2}{a}$, so that it also meets the axis of x in two points, b, a, equidistant from O. If α is numerically greater than $\frac{c^2}{a}$ the ordinates become imaginary, and if β is numerically greater than $\frac{c^2}{b}$ the abscissa becomes imaginary; therefore the curve is limited by the four points a, b, c, d, and touches the axes at those points. It consists, therefore, of four branches symmetrically situated as in the figure.

3. To determine the evolute of the rectangular hyperbola, its equation between the asymptotes being $xy = a^2$.

The equation of the evolute is

$$(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = \frac{a^{\frac{2}{3}}}{4^{\frac{1}{3}}}.$$

THEOREM.

(102.) Normals to the curve are tangents to the evolute.

Let the equations of the curve and of its evolute be

$$y = \mathrm{F}x \text{ and } \beta = f\alpha,$$

then differentiating the equation (2) p. 131, considering α, β as variables as well as x, y, we have

$$1 - \frac{d\alpha}{dx} + p''(y - \beta) + p'^2 - p'\frac{d\beta}{dx} = 0,$$

but (130)

$$y - \beta = -\frac{p'^2 + 1}{p''}.$$

Hence, by substitution,

$$\frac{d\alpha}{dx} + p'\frac{d\beta}{dx} = 0$$

$$\therefore \frac{d\beta}{dx} \div \frac{d\alpha}{dx} \text{ or } \frac{d\beta}{d\alpha} = -\frac{1}{p'} = \frac{\beta - y}{\alpha - x} \text{ (equa. 2, p. 131).}$$

Now $\dfrac{d\beta}{d\alpha}$ expresses the trigonometrical tangent of the angle between the axis of x and a linear tangent through any point (α, β) of the evolute, and $-\dfrac{1}{p'}$ expresses the trigonometrical tangent of the angle between the axis of x and a normal at any point (x, y) of the involute; but this normal necessarily passes through a point (α, β) of the evolute, and, therefore, in consequence of the above equality, it must coincide with the tangent at that point.

THEOREM.

(103.) The difference of any two radii of curvature is equal to the arc of the evolute comprehended between them.

Differentiating the equation

$$(y-\beta)^2+(x-\alpha)^2=r^2,$$

on the hypothesis that α is the independent variable, we have

$$-(y-\beta)\frac{d\beta}{d\alpha}-(x-\alpha)=r\frac{dr}{d\alpha};$$

but by last article

$$y-\beta=(x-\alpha)\frac{d\beta}{d\alpha}$$

$$\therefore (x-\alpha)^2\left(\frac{d\beta^2}{d\alpha^2}+1\right)=r^2\ldots(1),$$

and

$$-(x-\alpha)\left(\frac{d\beta^2}{d\alpha^2}+1\right)=r\frac{dr}{d\alpha}\ldots(2).$$

Dividing (2) by the square root of (1) we have

$$-\sqrt{\frac{d\beta^2}{d\alpha^2}+1}=\frac{dr}{d\alpha},$$

that is (86)

$$-\frac{ds}{d\alpha}=\frac{dr}{d\alpha}\ \therefore\ -s=r\pm\text{ a constant},$$

for otherwise there could not be $-\dfrac{ds}{d\alpha}=\dfrac{dr}{d\alpha}$.

Hence if r, r' be the radii of curvature of any two points, and s, s' the corresponding arcs of the evolute, then

$$\begin{array}{rcl} r\pm\text{const.} &=& -s \\ r'\pm\text{const.} &=& -s' \\ \hline r-r' &=& s'-s, \end{array}$$

so that the difference of the two radii is equal to the arc of the evolute comprehended between them; therefore, if a string fastened to one extremity of this arc be wrapped round it and continued in the direction of the tangent at the other extremity as far as the involute curve, the portion of the string thus coinciding with the tangent will by (102) be the radius of curvature at that point P of the involute curve which it meets, and, consequently, by the above property, if the string be now unwound, P will trace out the involute.

On Consecutive Lines and Curves.

(104.) Every equation between two variables may always be considered as the analytical representation of some plane curve, given in species by the degree of the equation, and determinable both in form and position by the constants which enter it, provided, these constants are fixed and determinate. If, however, the equation contains an arbitrary or indeterminate constant α, then, by assuming different values for α the equation will represent so many different curves varying in form and position, but all belonging to the same *family of curves.*

Now if we consider the form and position of one of these curves to be fixed by the condition $\alpha = \alpha'$, another, intersecting this in some point (x', y'), may be determined from a new condition $\alpha = \alpha' + h$; and if h be continually diminished, this latter curve will approach more and more closely to the fixed curve, and will at length coincide with it. During this approach, the point of intersection (x', y') necessarily varies, and becomes fixed in position only when the varying curve becomes coincident with the fixed curve. In this position the point is said to be the intersection of *consecutive curves*, so that what mathematicians call consecutive curves, are, in reality, coincident curves, and the point which has been denominated their point of intersection may be determined as follows:

(105.) Let

$$F(x, y, x') = 0 \dots (1)$$

represent any plane curve, x' being a parameter, and for any intersecting curve of the same family let x' become $x' + h$, then, since however numerous these intersecting curves may be, the x, y of the intersections belong also to the equation (1); it follows that as far as these points are concerned, the only quantity in equation (1) which varies is x', therefore, considering x, y as constants in reference to these points, we have, by Taylor's theorem,

$$F(x, y, x' + h) = F(x, y, x') + \frac{dF(x, y, x')}{dx'} h + \frac{d^2F(x, y, x')}{dx'^2} \frac{h^2}{1 \cdot 2} + \&c.$$

but $F(x, y, x') = 0$, therefore

$$\frac{\mathrm{F}(x, y, x' + h)}{h} = \frac{d\mathrm{F}(x, y, x')}{dx'} + \frac{d^2\mathrm{F}(x, y, x')}{dx'^2} \frac{h}{1 \cdot 2} + \&c.$$

hence, when the curves are consecutive, that is when $h = 0$, we have the following conditions, viz.

$$\left.\begin{array}{r} \mathrm{F}(x, y, x') = 0 \\ \dfrac{d\mathrm{F}(x, y, x')}{dx'} = 0 \end{array}\right\} \dots \dots (2)$$

to determine x and y.

Suppose, for example, it were required to determine the point of intersection of consecutive normals in any plane curve.

Representing the equation of the curve by

$$y' = \mathrm{F}x',$$

and any point in the normal by (x, y), we have for the equation of the normal

$$y - y' = -\frac{1}{p'}(x - x') \text{ or } (y - y')\,p' + x - x' = 0.$$

This corresponds to the first of equations (2), x' being the parameter; hence, differentiating with respect to x' of which y' is a function given by the equation of the curve, we have

$$(y - y')p'' - p'^2 - 1 = 0$$

$$\therefore y = y' + \frac{p'^2 + 1}{p''}$$

$$\therefore x = x' - \frac{p'(p'^2 + 1)}{p''}$$

hence (92) consecutive normals intersect at the centre of curvature.

(106.) If we eliminate the variable parameter x' by means of the equations (2), the resulting equation will belong to every point of intersection given by every curve of the family

$$\mathrm{F}(x, y, x, x') = 0 \dots \dots (1),$$

and its consecutive curve; for whatever value we suppose x' to take in the equations (2), the result of the elimination will obviously be always the same. Hence this resulting equation represents the locus of all the intersections, and we may show that at these same intersections this locus touches every individual curve in the family. The equation (1), where x' represents a function of x, y, determined by

the second of the conditions (2) in last article, is obviously the equation of the locus of which we are speaking, and the same equation, when x' takes all possible values from 0 to $\pm\,\infty$, furnishes the family of curves, which we are now to show are all touched by this locus. Taking any one of this family, and differentiating its equation (1), x' being constant, we have

$$du = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy = 0.$$

Differentiating also the equation (1) of the locus, x' being given by the second condition of (2) in last article, we have

$$du = \frac{du}{dx}\,dx + \frac{du}{dy}\,dy + \frac{du}{dx'}\,dx' = 0,$$

but by the condition just referred to $\frac{du}{dx'} = 0$ at the point where the curves whose equations we have just differentiated meet; hence, since at those points each of these equations give the same value for $\frac{dy}{dx}$, it follows that they have contact of the first order; we infer, therefore, that the equation (1), when x' is determined from the second of the conditions (2) last article, *represents a curve which touches and envelopes the entire family of curves represented by equation* (1), x' being any arbitrary constant. Thus, as we already know, the locus of the intersections of normals with their consecutive normals is a curve which touches them all at their points of intersection, being the evolute of the curve to which the normals belong.

The following examples will further illustrate this theory.

EXAMPLES.

(107.) 1. To determine the curve which touches an infinite series of equal circles, whose centres are all situated on the same circumference.

Let the equation of the fixed circle be

$$x'^2 + y'^2 = r'^2,$$

then, for the coordinates of the centre of any of the variable circles, the expressions will be

$$x' \text{ and } \sqrt{r'^2 - x'^2},$$

so that the general representation of these circles will be

$$(x - x')^2 + (y - \sqrt{r'^2 - x'^2})^2 - r^2 = 0 = u \dots (1),$$

x' being considered as an arbitrary constant. If, however, x' be considered not as an arbitrary constant, but as a function of x and y, fulfilling the condition $\frac{du}{dx'} = 0$, then, by the preceding theory, (1) will represent the curve which touches all the circles in those points where each is intersected by its consecutive circle. Hence, differentiating (1) with respect to x', we have

$$\frac{du}{dx'} = -(x - x') + \frac{x'y}{\sqrt{r'^2 - x'^2}} - x' = 0$$

$$\therefore -x\sqrt{r'^2 - x'^2} + x'y = 0$$

$$\therefore x' = \frac{r'x}{\sqrt{x^2 + y^2}}.$$

This, then, is the function of x, y, which, substituted for x', in (1), gives the equation of the locus sought. The result of this substitution is

$$x^2 + y^2 - 2r'\sqrt{x^2 + y^2} + r'^2 = r^2,$$

or, extracting the root of each side,

$$\sqrt{x^2 + y^2} = r' \pm r \therefore x^2 + y^2 = (r' \pm r)^2,$$

an equation representing *two* circles, whose radii are respectively $r' + r$ and $r' - r$. Hence the series of circles are touched and enveloped by two circular arcs, having these radii, and the same centre as the fixed circle.

2. Between the sides of a given angle are drawn an infinite number of straight lines, so that the triangles formed may all have the same surface, required the curve to which every one of these lines is a tangent.

Let the given angle be θ, and, taking its sides for axes, we have, for the equation of every variable line,

$$y = \alpha x + \beta \dots (1),$$

and, putting successively $y = 0$ and $x = 0$, the resulting expressions for x and y denote the sides of the variable triangle, including the

given angle, so that these sides are $-\dfrac{\beta}{\alpha}$ and β; hence, calling the constant surface s, we have

$$s = -\frac{\beta^2}{2\alpha}\sin.\theta \therefore \alpha = -\frac{\beta^2\sin.\theta}{2s};$$

hence the equation (1) is the same as

$$y = -\frac{\beta^2\sin.\theta}{2s}x + \beta \ . \ . \ . \ . \ (2),$$

where β is considered as an arbitrary constant. But if for this arbitrary constant we substitute the function of x, arising from the condition $\dfrac{dy}{d\beta} = 0$, then (2) will represent the locus of the intersections of each variable line, with its consecutive line, which locus touches them all. Differentiating them with regard to β, we have

$$-\frac{\beta\sin.\theta}{s}x + 1 = 0 \therefore \beta = \frac{s}{x\sin.\theta},$$

this substituted in (2) gives for the equation of the sought curve

$$y = -\frac{s}{2x^2\sin.\theta}x + \frac{s}{x\sin.\theta},$$

or rather

$$xy = \frac{s}{-2\sin.\theta};$$

hence the curve is an hyperbola, having the sides of the given angle for asymptotes.

3. The centres of an infinite number of equal circles are all situated on the same straight line: required the line which touches them all?

Ans. They are touched by two parallels to the line of centres.

4. From every point in a parabola lines are drawn, making the same angle with the diameter that the diameter makes with the tangent: required the line touching them all?

Ans. They are touched by a point, viz. the focus, in which therefore they all meet.

CHAPTER IV.

ON THE SINGULAR POINTS OF CURVES, AND ON CURVILINEAR ASYMPTOTES.

Multiple Points.

(108.) If several branches of a curve meet in one point, whether by intersecting or touching each other, that point is called a multiple point. In the former case the point is said to be of the first species, and in the latter of the second species, and we propose here to inquire how, by means of the equation of any curve, these points, if any, may be detected.

Multiple points of the first species. When the curve has multiple points of the first species, we readily arrive at the means of determining their position from the consideratioa that at such points there must be as many rectilinear tangents as there are touching branches, and, consequently, as many values for $\frac{dy}{dx}$, the tangent of the inclination of any tangent through the point (x, y) to the axis of x; so that the equation of the curve being freed from radicals and put under the form

$$\mathrm{F}(x, y) = 0,$$

its multiple points of the first species will all be given analytically by the equation

$$p' = -\frac{du}{dx} \div \frac{du}{dy} = \frac{0}{0},$$

so that no systems of values for x and y can belong to multiple points of the first species, but such as satisfy the conditions

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0,$$

as well as the equation of the curve. Having, therefore, determined all such systems of values by solving the two last equations, the true values of p' for each system will be ascertained by proceeding as in (41), and those systems only will belong to multiple points of the

first species that give multiple values to p'. Let us apply this to an example or two.

EXAMPLES.

(109.) 1. To determine whether the curve represented by the equation

$$ay^3 - x^3y - bx^3 = 0,$$

has any intersecting branches

$$\frac{du}{dx} = -3x^2(y+b), \frac{du}{dy} = 3ay^2 - x^3.$$

At the points where branches intersect we must have

$$3x^2(y+b) = 0, 3ay^2 - x^3 = 0$$
$$\therefore x = 0, y = 0$$

or

$$x = \sqrt[3]{3ab^2},\ y = -b;$$

this seeond system of coordinates do not satisfy the proposed equation, and therefore do not mark any point in the curve; the first system, which is admissible, shows that if there exist any multiple point it must be at the origin. Hence, to ascertain the true value of p' at this point, we have, by differentiating both numerator and denominator in the expression

$$[p'] = [\frac{3x^2(y+b)}{3ay^2 - x^3}] = \frac{0}{0}$$

$$= [\frac{6x(y+b) + 3x^2p'}{6ayp' - 3x^2}] = \frac{0}{0}$$

$$= [\frac{6(y+b) + 12p'x + 3x^2p''}{6ayp'' + 6ap'^2 - 6x}] = \frac{6b}{6a[p']^2} \therefore [p'] = \sqrt[3]{\frac{b}{a}};$$

therefore, as this has but one *real* value, the curve has no intersecting branches.

2. To determine whether the curve represented by the equation

$$x^4 + 2ax^2y - ay^3 = 0$$

has intersecting branches

$$\frac{du}{dx} = 4x(x^2 + ay) = 0, \frac{du}{dy} = a(2x^2 - 3y^2) = 0.$$

There is but one system of values that can satisfy these three equations, viz.

$$x = 0, y = 0,$$

so that if there are intersecting branches they must intersect at the origin. To determine, therefore, whether at this point p' has multiple values we have

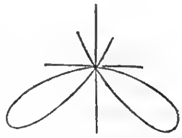
$$[p'] = [\frac{4x(x^2 + ay)}{a(3y^2 - 2x)^2}] = \frac{0}{0}$$

$$= [\frac{6x^2 + 2ay + 2axp'}{3ayp' - 2ax}] = \frac{0}{0}$$

$$= \frac{4a[p']}{3a[p']^2 - 2a}$$

$$\therefore 3a[p']^3 - 6a[p'] = 0$$

$$\therefore [p'] = 0 \text{ or } [p'] = \pm \sqrt{2};$$



hence *three* branches of the curve intersect at the origin; the tangent to one of them at that point is parallel to the axis of x, and the tangents to the other two are symmetrically situated with respect to the axis of y, since they are inclined to the axis of x, at angles whose tangents are $+\sqrt{2}$ and $-\sqrt{2}$.

(110.) Should the values of p' corresponding to any values of x and y, which satisfy the equation of the curve, be all imaginary, we must infer that, although such a system of values belong to a point of the locus, yet that point must be detached from the other points of the locus, for since, if the abscissa of this point be increased by h, the development of the ordinate will agree with Taylor's development, as far, at least, as the second term for all values of h, between some finite value and 0, it follows that all the corresponding ordinates between these limits must be imaginary, so that the proposed point is isolated, having no geometrical connexion with the curve, although its coordinates satisfy the equation. Such a point is called *a conjugate point*.

(111.) From what has now been said, it appears that, by having the equation of a plane curve given, those points in it where branches intersect, as also those which are entirely detached from the curve, although belonging to its equation, may always be determined by the

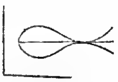
application of the differential calculus, and independently of all considerations about the failing cases of Taylor's theorem, except, indeed, those connected with the theory of vanishing fractions. We shall now seek the analytical indications of

Multiple Points of the Second Species.

(112.) The second species of multiple points, or those where branches of the curve touch each other, the differential calculus does not furnish the means of readily determining from the implicit equation of the curve. We know that at such a point, p' cannot admit of different values, since the branches have one common tangent; and we know, moreover, that if Taylor's theorem does not fail at that point, we shall, by successively differentiating, at length arrive at a coefficient which, being put under the form $\frac{0}{0}$, the different values will indicate so many different touching branches; for if no coefficient gave multiple values for the proposed coordinates x', y', then the ordinates corresponding to the abscissas between the limits x' and $x \pm h$, h being of some finite value, would each have but one value, and, therefore, different branches could not proceed from the point (x', y'). But we have no means of ascertaining *à priori* which of the coefficients furnishes the multiple value. When, however, the equation of the curve is explicit, then the multiple points of either species are very easily determined. Thus, if the equation of the curve be

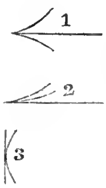
$$y = (x - a)^2 \sqrt{x - b} + c,$$

we at once see that $x = a$ destroys the radical in y and p', that re-appears in p''; therefore, at the point corresponding to this abscissa, there will be but one tangent, and yet two branches of the curve proceed from it on account of the double value of p''. Hence the point is a *double* point of the second species, the branches have contact of the first order, and, because $p' = 0$, the common tangent is parallel to the axis of the abscissas; if the radical had been of the third degree, the point corresponding to the same abscissa would have been a *triple* point, &c. It appears, therefore, that when the equation of the curve is solved for y, there will exist a multiple point, if in the expression for x a radical is multiplied by the factor $(x - a)^m$. If $m = 1$, the branches of the curve intersect at the point whose abscissa is $x = a$, because then p' at that point takes the same



values as the radical, but if $m > 1$ then the branches touch, because then the radical is destroyed in p' for $x = a$; in both cases the index of the radical will denote the number of branches which meet in the point. Such, therefore, are the geometrical significations of the cases discussed in (75) and (76).

Cusps, or Points of Regression.



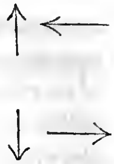
(113.) A *cusp* or *point of regression* is that particular kind of *double* point of the second species in which the two touching branches terminate, and through which they do not pass, so that on one side of such a point, viz. on that where the branches lie, the ordinate has a double value, and on the other side the contiguous ordinate has an imaginary value.

The cusp represented in the first figure, where the branches are one on each side of the common tangent, is called a cusp of the first kind, and that in the second figure, where the branches are both on one side, a cusp of the second kind.

(114.) It is obvious that cusps can exist only at those points, the particular coordinates of which cause Taylor's theorem to fail, for if Taylor's theorem did not fail at such a point, then the ordinates in the vicinity, corresponding both to $x + h$ and to $x - h$, would be both possible or impossible at the same time. We are not, however, to infer that when the adjacent ordinates are real on the one side of any point, and on the other side imaginary, that a cusp *necessarily* exists at that point, for it is plain that the same analytical indications are furnished by the point which limits *any* curve in the direction of the axis of x, or at which the tangent is perpendicular to that axis, as in the third figure. It becomes important, therefore, in seeking particular points of curves to be able to distinguish the point which limits the curve in the direction of the axes from cusps.

(115.) Now at the limits, the tangents to the curve are parallel to the axes, the limits are therefore determined by the equations $\frac{dy}{dx} = \infty$ and $\frac{dy}{dx} = 0$, and they fulfil, moreover, the following additional con-

ditions, viz. 1°, the ordinate or abscissa, whichever it may be, that is parallel to the tangent, immediately *beyond* the limit, must be imaginary; but if it be ascertained that this is not the case, the point is not a limit but a cusp of the first kind, posited as in the annexed figures, or else a point of inflexion; the latter when the contiguous ordinates are the one greater and the other less than that at the point. 2°, Besides the first condition there must exist also this, viz. that immediately *within* the limit the double ordinate or abscissa, whichever may be parallel to the tangent, must have one of its values greater and the other less than at the point, but if both are greater or both less the point is not a limit but a cusp of the second kind, posited as in the annexed figures. Hence, when the branches forming the cusp touch the abscissa or the ordinate of the point, they may be discovered by seeking among the values which satisfy the equations $\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = \infty$, those which do not fulfil both the foregoing conditions. Let us illustrate this by examples.



EXAMPLES.

(116.) 1. To determine whether the curve whose equation is

$$(y - b)^3 = (x - a)^2$$

has a cusp at the point where the tangent is parallel to the axis of y.

By differentiating

$$\frac{dy}{dx} = \frac{2}{3} \cdot \frac{x - a}{(y - b)^2}$$

this becomes infinite for $y = b$, therefore the point to be examined is (a, b). In order to this, substitute $a \pm h$ for x, in the proposed equation, and we have, for the contiguous ordinates,

$$y = b \pm h^{\frac{2}{3}},$$

which is not imaginary either for $+ h$ or $- h$; the point (a, b) is therefore a cusp of the first kind, and posited as in the figure, since the contiguous values of y are both greater than b.

2. To determine whether the curve whose equation is

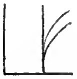
$$y - a = (x - b)^{\frac{1}{3}} + (x - b)^{\frac{3}{4}}$$

has a cusp at the point where the tangent is parallel to the axis of y.

Here the coefficient $\frac{dy}{dx}$ becomes infinite for $x = b$, therefore the point to be examined is (b, a). Substituting $b + h$ for x, we have

$$y = a + h^{\frac{1}{3}} + h^{\frac{3}{4}}.$$

For negative values of h this is imaginary, therefore the curve lies entirely to the right of the ordinate $y = a$, so that the condition 1° pertaining to a limit is fulfilled. To the right of this ordinate the two values of y, corresponding to a value of h, ever so small, are *both* greater than $y = a$, so that the condition 2° is not fulfilled, the point (b, a) is therefore a cusp of the second kind, and posited as in the cut.



3. To determine the point of the curve whose equation is

$$(y - a - x)^4 = (x - b)^3,$$

at which the tangent is parallel to the axis of y.

The differential coefficient becomes infinite for $x = b$, therefore the point to be examined is $(b, a + b)$. Substituting $b + h$ for x,

$$y = (a + b) + h^{\frac{3}{4}} + h,$$

negative values of h render this imaginary, therefore the condition 1° is fulfilled; positive values give two values for y, and as h may be taken so small that $h^{\frac{3}{4}}$ may exceed h, and since, moreover, the two values of $h^{\frac{3}{4}}$ are the one positive and the other negative, it follows that the real ordinate contiguous to the point has one value greater, and the other less, than that at the point of contact; hence the condition 2° is also fulfilled, and thus the point marks the limit of the curve, which, therefore, lies to the right of the ordinate, through $x = b$.

(117.) Having thus seen how to determine those cusps where the branches touch an ordinate or abscissa, we shall now seek how to discover those at which the tangent is oblique to the axes. The true development of the ordinate contiguous to such a cusp must be of the form

$$y' + \frac{dy'}{dx'}h + Ah^{\alpha} + Bh^{\beta} + \&c.$$

and the corresponding ordinate of the tangent will be

$$y' + \frac{dy'}{dx'} h ;$$

hence, subtracting this from the former, we have

$$\Delta = Ah^{\alpha} + \beta h^{\beta} + \&c.$$

(118.) Now in order that the point (x', y') may be a cusp, this difference for a small value of h must have two values, and to be a cusp of the first kind these two values must obviously have opposite signs; but since h may be so small that Ah^{α} may exceed the sum of all the following terms, h^{α} must have two opposite values; hence, α must be a fraction with an *even* denominator, and, conversely, if α be a fraction with an even denominator, the point (x', y') will be a cusp of the first kind. Hence, at such a point, $\frac{d^2y}{dx^2}$ is either 0 or ∞: 0 if $\beta > 2$, and ∞ if $\beta < 2$.

(119.) In order that the cusp may be of the second kind, both values of Δ must have the same sign there, for h^{α} cannot admit of opposite values of the same value of h, consequently α must in this case be either a whole number, or else a fraction with an *odd* denominator; and conversely, if α be either a whole number, or a fraction with an odd denominator, the point (x', y') will be a cusp of the second kind, provided, of course, that Δ has *two* values. The position of the branches will depend on the sign of A.

We shall now give an example or two.

(120.) 4. To determine whether the curve whose equation is

$$y = x \pm x^{\frac{9}{2}}$$

has a cusp.

Here y is possible for positive values of x, and imaginary for all negative values; hence there *may* be a cusp at the origin. To ascertain this, put h for x, in the equation, and we have, for the contiguous ordinate, the value

$$y = h \pm h^{\frac{9}{2}}.$$

The coefficient of h being $1 = \frac{dy'}{dx'}$ we see that the tangent to the curve at the origin is inclined at 45° to the axes, and, since $\frac{3}{2}$ has an even denominator the origin is a cusp of the first kind.

5. To determine whether the curve whose equation is

$$y - a = x + bx^2 + cx^{\frac{5}{2}}$$

has a cusp.

Here y is imaginary for all negative values of x, therefore the point $(0, a)$ *may* be a cusp. Substituting h for x, we have

$$y = a + h + bh^2 + ch^{\frac{5}{2}}.$$

As before, the tangent is inclined at 45° to the axes, and, since the exponent of the third term is a whole number, and the whole expression admits of two values, in consequence of the even root $h^{\frac{5}{2}}$, it follows that the proposed point is a cusp of the second kind. The branches are situated to the right of the axis of y, because h must be positive, and they are above the tangent because bh^2 is positive.

6. To determine whether the curve whose equation is

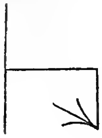
$$(2y + x + 1)^2 = 2\,(1 - x)^5$$

has a cusp.

Here values of x greater than 1 are obviously inadmissible, and to this value of x corresponds $y = -1$; hence the point having these coordinates *may* be a cusp. Substituting $1 + h$ for x, we have

$$y = -1 + \tfrac{1}{2}\,h + h^{\frac{5}{2}}$$

therefore the tangent to the curve at the proposed point has the trigonometrical tangent of its inclination to the axis of x equal to $\frac{1}{2}$, and



since the fraction $\frac{5}{2}$ has an even denominator, the point is a cusp of the first kind. Because h is negative, the branches are to the left of the ordinate to the point which is below the axis of x, because this ordinate is negative.

Points of Inflexion.

(121.) Points of inflexion have been defined at (92), and we have

there shown that a point of this kind always exists when its abscissa causes all the differential coefficients to vanish between the first and the nth, provided the nth be odd and becomes neither 0 nor ∞. The simplest indication therefore of a point of inflexion is $[\frac{d^2y}{dx^2}] = 0$, and $[\frac{d^3y}{dx^3}]$ neither 0 nor ∞; such indications, however, cannot be furnished by any point at which the tangent is parallel to the axis of y, since in this case $[\frac{dy}{dx}]$ and all the following coefficients become infinite. Neither can these indications take place at any point, for which Taylor's theorem fails after the third term. It becomes, therefore, of consequence, in examining particular points of a curve, to be able to detect the existence of points of inflexion by some general method. independently of the differential coefficients beyond the first. The only general method of doing this is that which we have already employed for the discovery of cusps, and which consists simply in examining the course of the curve in the immediate vicinity and on each side the point in question. Points of inflexion are somewhat similar to cusps, each having some of the analytical characteristics common to both, and to the limiting points of curves as already hinted at in (114). But the characteristic property of a point of inflection is, that the adjacent ordinates on each side are the one greater and the other less than the ordinate at the point. This peculiarity distinguishes a point of inflexion from a limit, inasmuch as at a limit the ordinate immediately beyond is imaginary; and it distinguishes it from a cusp of the first kind, inasmuch as at such a cusp the adjacent ordinates are either both greater or both less than at the point,

or else, as is the case when the tangent at the point is oblique to the axes, one of these ordinates is imaginary, the other double. We have then first to ascertain at what points of the curve inflexions *may* exist, or to find what points are given by the conditions

$$\frac{d^2y}{dx^2} = \frac{P}{Q} = 0 \text{ or } \infty,$$

or, which is the same thing, what points are given by the separate conditions.

$$P = 0, \; Q = 0,$$

we are then, by examining the course of the curve in the vicinity of each point, to determine to which of them really belongs the characteristic of an inflexion.

Thus the means of distinguishing points of inflexion being sufficiently clear, we shall proceed to a few examples.

EXAMPLES.

1. To determine whether the curve whose equatio n

$$y = b + (x - a)^3$$

has a point of inflexion where the tangent is parallel to the axis of x.

Here

$$p' = 3\,(x - a)^2,$$

and when the tangent is parallel to the axis of x, $p' = 0$, $\therefore x = a$ and $y = b$, at the proposed point. In the vicinity $x = a + h$,

$$\therefore y = b + h^3,$$

which is *greater* than b, the ordinate of the point when h is positive, and less when h is negative; the point (a, b) is therefore a point of inflexion.

2. To determine whether the curve whose equation is $y^3 = x^5$ or $y = x^{\frac{5}{3}}$ has an inflexion at any point.

$$p' = \tfrac{5}{3}\,x^{\frac{2}{3}}, \; p'' = \tfrac{2}{3} \cdot \tfrac{5}{3}\,x^{-\frac{1}{3}}$$

this becomes ∞ for $x = 0$, therefore a point of inflexion *may* exist at the origin. Putting h for x we have

$$y = h^{\frac{5}{3}},$$

which is greater than 0, the ordinate of the point, when h is positive, and less when h is negative; hence there is an inflexion at the origin. Also the equation of the tangent being $y = \frac{5}{3}\,x^{\frac{2}{3}}$, the ordinates corresponding to $x = \pm h$ are both less than those given by the above equation; hence the curve lies above the tangent to the right of the origin, and below it to the left, as in the figure.

3. To determine whether the curve whose equation is

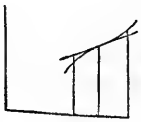
$$y - x = (x - a)^{\frac{5}{3}}$$

has a point of inflexion

$$p' = 1 + \tfrac{5}{3}(x - a)^{\frac{2}{3}},\ p'' = \tfrac{2}{3}\cdot\tfrac{5}{3}(x - a)^{-\frac{1}{3}}$$

this becomes infinite for $x = a$, therefore a point of inflexion *may* exist at the point (a, a). In the vicinity of this point $x = a + h$,

$$\therefore y = a + h + h^{\frac{5}{3}},$$



which is greater than a when h is positive, and less when h is negative; hence $(a, a,)$ is a point of inflexion. As the corresponding ordinates of the tangent $y = a \pm h$, one, viz. $y = a + h$, is less than that of the curve, and the other greater; hence the curve bends, as in the figure.

On Curvilinear Asymptotes.

(122.) Two plane curves, having infinite branches, are said to be asymptotes to each other, when they approach the closer to each other as the branches are prolonged, but meet only at an infinite distance.*

Hence, since the expression for the difference of the ordinates corresponding to the same abscissa in two such curves becomes less as the abscissa becomes greater, and finally becomes 0, when the abscissa becomes ∞, it follows that that expression can contain none but *negative* powers of x, without the addition of any constant quantity. For, if a positive power of x entered the expression for the difference, that expression would become not 0 but ∞, when $x = \infty$; and, if there were a quantity independent of x, the difference would be reduced to this quantity, and not to 0, for $x = 0$. Hence two curves are asymptotes to each other, when the general expression for the difference of the ordinates corresponding to the same abscissa is

$$\Delta = A'x^{-\alpha} + B'x^{-\beta} + C'x^{-\gamma} + \&c. \ldots . (1),$$

or when the general expression for the difference of the abscissas cor-

* Spirals meet their asymptotic circles only after an infinite number of revolutions; these we do not consider here, having examined them at (85).

responding to the same ordinate is

$$\Delta = A'y^{-\alpha} + B'y^{-\beta} + C'y^{-\gamma} + \&c. \quad . \quad . \quad . \quad . \quad (2),$$

and conversely, when the curves are asymptotes to each other; one or both these forms must have place.

If for one of the curves whose corresponding ordinates are supposed to give the difference (1) there be substituted another, which would reduce that difference to

$$B'x^{-\beta} + C'x^{-\gamma} + \&c.$$

this new curve would be an asymptote to both, and would obviously, throughout its course, continually approach nearer to that which it has been compared to, than the one for which we have substituted it does. In like manner, if a third curve would further reduce the difference (1) to

$$C'x^{-\gamma} + \&c.$$

this third curve would approach the first still nearer, and all the four would be asymptotes to each other. It appears, therefore, that every curve of which the ordinate may be expanded into an expression of the form

$$y = Ax^a + Bx^b + \quad . \quad . \quad . \quad . \quad A'x^{-\alpha} + B'x^{-\beta} + \&c. \quad . \quad . \quad . \quad . \quad (3).$$

admits of an infinite number of asymptotes.

Since the general expression for the ordinate of a straight line is $y = Ax + B$, for the difference between this ordinate and that of a curve at the point whose abscissa is x, to have the form (1), the equation of the curve must be

$$y = Ax + B + A'x^{-\alpha} + B'x^{-\beta} + \&c. \quad . \quad . \quad . \quad . \quad (4),$$

this equation, therefore, comprehends all the curves that have a rectilinear asymptote, and among them the common hyperbola, whose equation is

$$y = \pm \frac{B}{A}(x^2 - A^2)^{\frac{1}{2}} = \mp \frac{B}{A}x \mp \tfrac{1}{2}ABx^{-1} + \&c.$$

The curves included in the equation (4) are therefore called *hyperbolic curves.*

The other curves comprised in the more general equation (3), not admitting of a rectilinear asymptote, are called *parabolic curves.*

The common hyperbola we see by the above equation admits of the two rectilinear asymptotes $y = \pm \frac{B}{A} x$, and of an infinite number of hyperbolic asymptotes.

As an example of this method of discovering rectilinear and curvilinear asymptotes, let the equation

$$my^3 - xy^3 = mx^3$$

be proposed. The development of y in a series of descending powers of x is (Ex. 9, p. 52,)

$$y = -m - \frac{m^4}{x^3} - \&c.$$

therefore the curve has one rectilinear asymptote, parallel to the axis of x, its equation being $y = -m$; the hyperbolic asymptote next to this, and which lies closer to the curve, is of the fourth order, its equation being

$$yx^3 + mx^3 + m^4 = 0.$$

Again, let the equation of the proposed curve be

$$y = \frac{b}{(x^2 - a^2)^{\frac{1}{2}}}$$

$$= bx^{-1} + \&c. \quad . \quad . \quad . \quad . \quad (1),$$

also, since

$$x^2 - a^2 = \frac{b^2}{y^2} \therefore x = a + \tfrac{1}{2} \cdot \frac{b^2}{a} y^{-2} + \&c. \quad . \quad . \quad . \quad . \quad (2).$$

From (1) it appears that the curve has a rectilinear asymptote, coincident with the axis of x, its equation being $y = 0$; the hyperbola whose asymptotes coincide with the axes is also an asymptote, its equation being $xy = b$. From (2) it appears that the curve has another rectilinear asymptote, parallel to the axis of y, its equation being $x = a$; the hyperbola next to this is of the third order. If we consider the radical, in the proposed equation, to admit of either a positive or a negative value, then there will be two rectilinear asymptotes, parallel to the axis of y and equidistant from it, as also two hyperbolic asymptotes, symmetrically situated between the axes.

SECTION III.

ON THE GENERAL THEORY OF CURVE SURFACES AND OF CURVES OF DOUBLE CURVATURE.

CHAPTER I.

ON TANGENT AND NORMAL PLANES.

PROBLEM I.

(123.) To determine the equation of the tangent plane at any point on a curve surface.

Let $(x', y', z',)$ represent any point on a curve surface of which the equation is

$$z = \mathrm{F}\,(x, y),$$

then the tangent plane will obviously be determined, when two linear tangents through this point are determined. Let us then consider, for greater simplicity, the two linear tangents respectively parallel to the planes of xz, zy; their equations are

$$\left.\begin{array}{r} z - z' = a\,(x - x') \\ y = y' \end{array}\right\} \quad \ldots\ldots \quad (1),$$

and

$$\left.\begin{array}{r} z - z' = b\,(y - y') \\ x = x' \end{array}\right\} \quad \ldots\ldots \quad (2),$$

But since these are tangents to the plane curves, which are the sections through $(x', y', z',)$ parallel to the planes of xz, zy, therefore (77)

$$a = \frac{dz'}{dx'} = p',\ b = \frac{dz'}{dy'} = q'.$$

Moreover the traces of the plane through the lines (1), (2), upon the planes of xz, zy, being parallel to the lines themselves, a and b must be the same in the traces as in these lines, and since they are the

same in the plane as in its traces, it follows that the equation of this plane must be

$$z - z' = p'(x - x') + q'(y - y') \quad \ldots \quad (3),$$

in which the partial differential coefficients p', q', express the trigonometrical tangents of the inclinations of the vertical traces to the axes of x and y respectively.

For the angle which the horizontal trace makes with the axis of x we have, by putting $z = 0$, in (3),

$$\text{tan. inc.} \frac{p'}{q'}.$$

(124.) If the equation of the surface is given under the form

$$u = \mathrm{F}(x, y, z,) = 0 \quad \ldots \quad (4),$$

then the expressions for the total differential coefficients derived from u, considered as a function, first of the single variable x, and then of the single variable y, are (57)

$$\left\{\frac{du}{dx}\right\} = \frac{du}{dx} + \frac{du}{dz} p' = 0$$

$$\left\{\frac{du}{dy}\right\} = \frac{du}{dy} + \frac{du}{dz} q' = 0,$$

from which we get the values

$$p' = -\frac{\frac{du}{dx}}{\frac{du}{dz}}, \quad q' = -\frac{\frac{du}{dy}}{\frac{du}{dz}}$$

hence, by substituting these expressions in (3), the equation for the tangent plane becomes

$$(z - z')\frac{du}{dz} + (x - x')\frac{du}{dx} + (y - y')\frac{du}{dy} = 0 \quad \ldots \quad (5).$$

PROBLEM II.

(125.) To determine the equation of the normal line at any point of a curve surface.

We have here merely to express the equation of a straight line, perpendicular to the plane (3), and passing through the point of contact $(x', y', z'.)$

Now the projections of this line must be perpendicular to the traces of the tangent plane, or to the lines (1), (2,) hence the equations of these projections must be

$$\left.\begin{array}{l} x - x' + p'(z - z') = 0 \\ y - y' + q'(z - z') = 0 \end{array}\right\}$$

which together, therefore, represent the normal.

(126.) If we represent by α, β, γ, the inclinations of this line to the axes of x, y, z, respectively, then (*Anal. Geom.*)

$$\cos.\ \alpha = \frac{-p'}{\sqrt{p'^2 + q'^2 + 1}}$$

$$\cos.\ \beta = \frac{-q'}{\sqrt{p'^2 + q'^2 + 1}}$$

$$\cos.\ \gamma = \frac{1}{\sqrt{p'^2 + q'^2 + 1}}$$

(127.) If the equation of the surface be given under the form (4), last problem, then, in these expressions for the inclinations, we must, instead of p' and q', write their values as before determined from that equation. If, for brevity, we put

$$v = \frac{1}{\sqrt{\frac{du^2}{dx^2} + \frac{du^2}{dy^2} + \frac{du^2}{dz^2}}}$$

the expressions for the cosines will then be

$$\cos.\ \alpha = v\frac{du}{dx}, \cos.\ \beta = v\frac{du}{dy}, \cos.\ \gamma = v\frac{du}{dz}.$$

As every plane which contains the normal line must be perpendicular to the tangent plane, it is obvious that there exists an infinite number of normal planes to any point of a surface.

PROBLEM III.

(128.) To determine the equation of the tangent line to any point of a curve of double curvature.

We have already indicated (*Anal. Geom.*) how this equation is to be determined:

Let

$$y = fx, z = Fx \quad . \; . \; . \; . \quad (1)$$

be the equations of the projections of the proposed curve, on the planes of xy, xz, and let $(x', y', z',)$ be the point to which the linear tangent is to be drawn, which point will be projected into (x', y') and $(x', z',)$ on the plane curves (1), therefore tangents through them to these plane curves will be represented by the two equations

$$\left.\begin{aligned} y - y' &= p' (x - x') \\ z - y' &= q' (x - x') \end{aligned}\right\} \quad . \; . \; . \; . \quad (2),$$

these, therefore, together represent the required tangent in space.

PROBLEM IV.

(129.) To determine the equation of the normal plane at any point in a curve of double curvature.

The equation of any plane passing through a proposed point is (*Anal. Geom.*)

$$A (x - x') + B (y - y') + C (z - z') = 0 \quad . \; . \; . \; . \quad (1),$$

and for the traces of this plane on the planes of xy, xz, we have, by putting in succession $z = 0$, $y = 0$, the equations

$$y - y' = -\frac{A}{B} (x - x') + \frac{C}{B} z'$$

$$z - z' = -\frac{A}{C} (x - x') + \frac{B}{C} y',$$

but since these two traces are respectively perpendicular to those marked (2), last problem,

$$\therefore \frac{B}{A} = p', \frac{C}{A} = q';$$

hence the equation (1) becomes

$$x - x' + p' (y - y') + q' (z - z') = 0 \quad . \; . \; . \; . \quad (2),$$

which represents the normal plane sought.

CHAPTER II.

ON CYLINDRICAL SURFACES, CONICAL SURFACES, AND SURFACES OF REVOLUTION.

(130.) THESE surfaces have been considered in the *Analytical Geometry*, and the general equations of the two first classes have been deduced, on the hypothesis that the directrix is always a plane curve. We shall now suppose the directrix to be any curve situated in space, and investigate the differential equations of these surfaces, as also of surfaces of revolution in general.

Conical and Cylindrical Surfaces.

PROBLEM I.

To determine the equation of cylindrical surfaces in general.

Let the equations of the generating straight line be

$$\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\} \therefore \left\{\begin{array}{l} \alpha = x - az \\ \beta = y - bz \end{array}\right. \quad \ldots \; (1),$$

and the equations of any curve in space considered as the directrix,

$$\mathrm{F}\,(x, y, z) = 0, f\,(x, y, z) = 0 \quad \ldots \; (2).$$

Now for every point in this directrix, all these equations exist simultaneously; moreover, the constants a, b, are fixed, since the inclination of the generating line does not vary, but the constants α, β, are not fixed, since the *position* of the generating lines does vary. If, then, we eliminate x, y, z, from the above equations, there will enter, in the resulting equation, only the constants a, b, and the indeterminates α, β, hence, solving this equation, for β we shall get a result of the form $\beta = \varphi\alpha$; consequently, if we now substitute in this the values of α and β given above, in terms of x, y, z, we shall have this general relation among these variables, viz.

$$y - bz = \varphi\,(x - az) = 0 \quad \ldots \; (3),$$

which is the equation of cylindrical surfaces in general, the function φ depending entirely on the nature of the directrix.

(131.) Now, by differentiation, this function may be eliminated (58), hence,

$$\frac{-bp'}{1-bq'} = \frac{1-ap'}{-aq'},$$

$$\therefore ap' + bq' = 1 \text{ or } a\frac{dz}{dx} + b\,\frac{dz}{dy} = 1 \ldots . (4),$$

which is the general differential equation of cylindrical surfaces.

(132.) The same equation may be immediately deduced from the general equation of a tangent plane, to the cylindrical surface. Thus, the equation of any tangent plane, through a point (x', y', z') being

$$z - z' = p'(x - x') + q'(y - y'),$$

the condition necessary for it to be *always* tangent to the cylinder on which this point is situated, is merely that it may be always parallel to its generatrix (1), and this condition, expressed analytically, is (*Anal. Geom.*)

$$ap' + bq' - 1 = 0 \ldots . (4),$$

this is, therefore, the relation which must have place between the partial differential coefficients derived from the equation of the surface, in order that that surface may be cylindrical, and it agrees with the relation before established.

If, in this equation, we write for p', q', their values deduced from the equation $u = 0$ of any cylindrical surface, as exhibited in (124), it becomes

$$a\frac{du}{dx} + b\,\frac{du}{dy} + \frac{du}{dz} = 0 \quad . \quad . \quad . \quad . \quad (5).$$

PROBLEM II.

(133.) Given the equation of the generatrix, to determine the cylindrical surface which envelopes a given curve surface.

Since the cylinder envelopes the given surface, the curve of contact is common to both, therefore every tangent plane to the cylinder touches the enveloped surface in that curve. The equation of any of these tangent planes is

$$z - z' = p'(x - x') + q'(y - y'),$$

whether p' and q' be derived from the equation of the surface, and take those particular values which restrict them to the curve of con-

tact, or whether p' and q' be derived from the equation of the cylinder, and preserve their general values, because in the one case the contact of each tangent is confined to a point in the curve of contact, and in the other case the contact extends along the whole length of the cylinder. Hence, for the curve of contact, the condition (5) must have place, as well as for the entire surface of the cylinder. The mode of solution is, therefore, obvious; we must deduce p' and q' from the equation of the given surface, and substitute them in (5), the result combined with the equation of the given surface, will obviously represent the curve for which p' and q' are common to both surfaces; that is to say, we shall thus have the equations of the directrix, and that of the generatrix being also given, the particular cylindrical surface becomes determined.

(134.) If the proposed curve surface be of the second order, then the equation (5) will necessarily be of the first degree in x, y, z, and will, therefore, represent a plane; so that the combination of this, with the equation of any surface, must necessarily represent a plane section of that surface; we infer, therefore, that if any cylindrical surface circumscribe a surface of the second order, the curve of contact will always be a plane curve, and consequently of the second order, and therefore the cylinder itself must be of the second order.

PROBLEM III.

(135.) To determine the general equation of conical surfaces.

Let $(x', y', z',)$ be the vertex of the conical surface, then since the generatrix always passes through this point, its equations, in any position, will be

$$\left.\begin{array}{l} x - x' = a\,(z - z') \\ y - y' = b\,(z - z') \end{array}\right\} \quad \ldots . \; (1),$$

$$\therefore a = \frac{x - x'}{z - z'},\; b = \frac{y - y'}{z - z'}.$$

Also let the equations of the directrix be

$$\mathrm{F}\,(x, y, z) = 0, f\,(x, y, z) = 0 \quad \ldots . \; (2),$$

then, since for every point in this line, the equations (1) and (2) exist together, we may eliminate the variables x, y, z; the result will be an equation, containing the fixed constants x', y', z' and the indeterminates a, b; therefore, solving this equation for b, we shall have $b =$

φa. Hence, substituting for a, b, their values in terms of x, y, we have

$$\frac{y - y'}{z - z'} = \varphi\left(\frac{x - x'}{z - z'}\right),$$

for the equation of conical surfaces in general, the function φ depending entirely on the directrix.

(136.) Eliminating the function φ, by differentiating each member of this equation with respect to x and y, and dividing the results as in (58), we have

$$\frac{(y - y')\, p'}{z - z' - (y - y')\, q'} = \frac{z - z' - (x - x')\, p'}{(x - x')\, q'},$$

which reduces to

$$z - z' = p'\,(x - x') + q'\,(y - y'),$$

the differential equation of conical surfaces in general.

(137.) This same equation, like that of cylindrical surfaces, may be obtained more readily by the consideration of the tangent plane, which, as it always passes through the vertex (x', y', z',) is, in every position, represented by the equation

$$z - z' = p'\,(x - x') + q'\,(y - y'),$$

this relation, therefore, must exist between the partial differential coefficients p', q', for every point of the surface, in order that it may be conical.

As in Problem I. if for p', q', we substitute their values derived from the implicit equation of any conical surface, the differential equation becomes

$$(x - x')\,\frac{du}{dx} + (y - y')\,\frac{du}{dy} + (z - z')\,\frac{du}{dz} = 0.$$

PROBLEM IV.

(138.) Given the position of the vertex, to determine the equation of the conical surface that envelopes a given curve surface.

Since the cone envelopes the proposed surface, the curve of contact is common to both, so that the tangent planes to the cone touch also the given surface, according to this curve. The equation, therefore, of the tangent plane

$$z - z' = p'\,(x - x') + q'\,(y - y') \quad . \quad . \quad . \quad . \quad (1),$$

holds equally for any point on the conical surface, and for any point in the curve of contact. Hence, if the values of p', q', be derived from the equation of the given surface, and substituted in (1), this, combined with the equation of the given surface, must represent the curve common to both surfaces, that is, the directrix of the cone. Therefore, the vertex and directrix being known, the equation of the required conical surface becomes determinable.

(139.) If the given curve surface be of the second order, the equation (1) will be also of the second order; but, nevertheless, the combination of these two equations will be that of a plane, for a surface of the second order may be generally represented by the equation

$$\left.\begin{array}{l} Ax^2 + By^2 \ \ + Cz^2 \\ + 2Dyz + 2Exz + 2Fxy \\ + 2Gy \ \ + 2Hy \ \ + 2Jz \end{array}\right\} = K \ldots . (2),$$

which gives

$$\frac{dz}{dx} = -\frac{Ax + Fy + Ez + G}{Ex + Dy + Cz + J}$$

$$\frac{dz}{dy} = -\frac{Fx + By + Dz + H}{Ex + Dy + Cz + J},$$

substituting these values for p' and q', in the equation (1), and subtracting from the result the equation (2), we have,

$$\left.\begin{array}{l} (Ax' + Fy' + Ez' + G)\,x \\ + (Fx' + By' + Dz' + H)\,y \\ + (Ex' + Dy' + Cz' + J)\ z \\ + \ Gx' + Hy' + Jz' + K \end{array}\right\} = 0 \ldots . (3),$$

which is the equation of a plane; therefore, the conical surface which circumscribes a surface of the second order, must itself be also of the second order.

(140.) The above proof is from *Monge* (*Application de l'Analyse à la Géométrie*), but it may be rendered much more concise, by assuming the axes of reference so as to give the general equation of the surface a simpler form. Thus, let the axes of x pass through the centre, if the surface have a centre, or be parallel to its diameters if it have not, and let the other two axes be parallel to the conjugates to this, the form of the equation will then be

$$Az^2 + By^2 + Cx^2 + 2Fx = G \ldots . (4),$$

$$\therefore \frac{dz}{dx} = -\frac{\mathrm{F} + \mathrm{C}x}{\mathrm{A}z}, \frac{dz}{dy} = -\frac{\mathrm{B}y}{\mathrm{A}z}.$$

These values, substituted for p' and q' in (1), convert that equation into

$$z - z' + \frac{(\mathrm{F} + \mathrm{C}x)(x - x')}{\mathrm{A}z} + \frac{\mathrm{B}y(y - y')}{\mathrm{A}z} = 0,$$

or

$$\mathrm{A}z^2 + \mathrm{B}y^2 + \mathrm{C}x^2 + \mathrm{F}x - \mathrm{A}z'z - \mathrm{B}y'y - \mathrm{C}x'x - \mathrm{F}x' = 0.$$

The difference between this, and (4), is

$$\mathrm{A}z'z + \mathrm{B}y'y + \mathrm{C}x'x + \mathrm{F}x + \mathrm{F}x' = \mathrm{G},$$

the equation of a plane.

(141.) Referring again to *Monge's* process, we may remark, that if we accent the constants in the general equation (2), it may be taken as the representative of another surface of the second order, for which the plane of contact with a circumscribing cone, whose summit coincides with that of the former cone, will be represented by the equation

$$\left.\begin{array}{l}(\mathrm{A}x' + \mathrm{F}y' + \mathrm{E}z' + \mathrm{G}')\,x \\ + (\mathrm{F}x' + \mathrm{B}y' + \mathrm{D}z' + \mathrm{H}')\,y \\ + (\mathrm{E}x' + \mathrm{D}y' + \mathrm{C}z' + \mathrm{J}')\,z \\ + \ \mathrm{G}x' + \mathrm{H}y' + \mathrm{J}z' + \mathrm{K}')\end{array}\right\} = 0 \quad . \ . \ . \ . \quad (5).$$

Now, although equation (3) be multiplied by an indeterminate constant, p, the result will still represent the same plane, and this plane will obviously be identical to that represented by (5), provided the coefficients of the variables x, y, z, are the same in both equations, that is to say, provided we have the conditions

$$\begin{array}{l}p\,(\mathrm{A}x' + \mathrm{F}y' + \mathrm{E}z' + \mathrm{G}) = \mathrm{A}x' + \mathrm{F}y' + \mathrm{E}z' + \mathrm{G}' \\ p\,(\mathrm{F}x' + \mathrm{B}y' + \mathrm{D}z' + \mathrm{H}) = \mathrm{F}x' + \mathrm{B}y' + \mathrm{D}z' + \mathrm{H}' \\ p\,(\mathrm{E}x' + \mathrm{D}y' + \mathrm{C}z' + \mathrm{J}) = \mathrm{E}x' + \mathrm{D}y' + \mathrm{C}z' + \mathrm{J}' \\ p\,(\mathrm{G}x' + \mathrm{H}y' + \mathrm{J}z' + \mathrm{K}) = \mathrm{G}x' + \mathrm{H}y' + \mathrm{J}z' + \mathrm{K}'.\end{array}$$

As, therefore, the four quantities x', y', z', p, are arbitrary they may be determined so that these conditions shall be fulfilled, the four equations being just sufficient to fix the values of these four quantities, and as each of them enters only in the first degree, they will each have but one value. It follows, therefore, that there is a certain point, and only one, from which, as a vertex, if tangent cones be

drawn to two given surfaces of the second order, their planes of contact shall coincide The common vertex will be at the intersection of those diameters to each of which the plane of contact is conjugate; since it has been shown above, that the vertex of the tangent cone is always situated on that diameter of the surface, to which the plane of contact is conjugate.*

(142.) We may here observe, that as we have not fixed the origin of the axes to any particular point on the diameter which has been taken for the axis of x, nor, indeed, the diameter itself, we may consider the diameter to be that passing through the vertex $(x', y', z',)$ of the cone, and this point to be the origin, in which case x', y', z', will each be 0, and the equation of the plane through the curve of contact, will then be simply

$$\mathrm{F}x = \mathrm{G} \therefore x = \frac{\mathrm{G}}{\mathrm{F}},$$

hence, the plane through the curve of contact, is conjugate to the diameter through the vertex of the cone. If this vertex be supposed infinitely distant, the same result will belong to the circumscribing cylinder, viz. that the plane of the curve of contact, is conjugate to the diameter parallel to the generatrix of the cylinder.

Surfaces of Revolution.

(143.) The surfaces of revolution, considered in the Analytical Geometry, comprise those only in which the revolving curve is always situated in the plane of the fixed axis. We shall here treat of surfaces of revolution in general, the revolving curve being any how situated with respect to the axes. Sections of the surface, in the plane of the axis, are called *meridians*.

PROBLEM V.

(144.) To determine the equation of surfaces of revolution in general.

Let the equations of the generating curve be

$$\mathrm{F}(x, y, z) = 0, f(x, y, z) = 0 \quad . \quad . \quad . \quad . \quad (1),$$

* For these, and other kindred properties, the student is referred to *Mr. Davies's* paper on *Geometry of Three Dimensions*, in *Leybourn's Repository*, vol. 5.

and those of the fixed axis

$$\left.\begin{array}{l} x = az + \alpha \\ y = bz + \beta \end{array}\right\} \quad \ldots\ldots (2),$$

then, since the characteristic property of surfaces of revolution is, that every section perpendicular to the fixed axis is a circle, we shall have first to determine a plane perpendicular to the line (2), and then to express the condition that this plane, combined with the surface, always represents a circle whose centre is on (2). Now the equation of the required plane is (*Anal. Geom.*)

$$z + ax + by = c \quad \ldots . (3),$$

and the condition is, that it must give the same section as if it were to cut a sphere, whose centre we may fix at pleasure, but whose radius will vary with the section, that is, it will depend upon c in equa. (3). Assuming the centre of this sphere at the point where the line (2) pierces the plane of xy, its equation will be (*Anal. Geom.*)

$$(x - \alpha)^2 + (y - \beta)^2 + z^2 = r^2 \quad \ldots . (4).$$

Hence, supposing r to be the proper function of c, the equations (1), (3), (4), must all have place together; hence we may eliminate x, y, z, and thus determine what the relation between r and c must necessarily be, to render these equations coexistent. The result of the elimination will obviously lead to $c = \varphi r^2$, hence, substituting for c and r their values in terms of the variables, we have, finally,

$$z + ax + by = \varphi \{(x - \alpha)^2 + (y - \beta)^2 + z^2\} \quad \ldots . (5),$$

for the relation which must always exist among the coordinates of every point, in every circular section. This, therefore, is the equation of surfaces of revolution in general.

(145.) If the fixed axis be taken for the axis of z, then a, α; b, β, are each 0, therefore, in this case, the general equation becomes

$$z = \varphi (x^2 + y^2 + z^2) \quad \ldots . (6),$$

which, solved for z, takes the form

$$z = \psi (x^2 + y^2) \quad \ldots . (7).$$

(146.) There is one case of this general problem, viz. that where the generatrix is a straight line, revolving round the axis of z, but not in the same plane with it, that deserves particular notice.

Let us take, for axis of x, the shortest distance between the axis of

z and the generating line; then this axis will be perpendicular to both (65), the equations, therefore, of the line will be

$$x = \pm\, a, y = \pm\, bz,$$

also, for any variable section perpendicular to the axis of z

$$z = c, x^2 + y^3 + z^2 = r^2.$$

Eliminating x, y, z, we have

$$a^2 + b^2c^2 + c^2 = r^2,$$

for c and r^2 putting their values above, we have

$$a^2 + b^2c^2 = x^2 + y^2.$$

By putting successively $x = 0$, $x = 0$ in this equation, the resulting forms belong to hyperbolas, hence the surface is the *hyperboloid of revolution of a single sheet.* The equation of the hyperbola corresponding to $x = 0$ is

$$z = \frac{y}{b}\sqrt{1 + \frac{a^2}{y^2}},$$

so that $y = \pm\, bz$ is the equation of the asymptotes, (see *Anal. Geom.*) hence the generating straight line, in its first position, is in a plane with and parallel to one or other of the asymptotes of that hyperbola in its first position, which would generate by revolving round the axis of z, the same surface as the line; these two lines, therefore, continue parallel during the revolution of both; the one, viz. the asymptote, generating the conical surface asymptotic to the hyperboloid generated by the other line, viz. the line

$$x = \pm\, a, y = bz,$$

or

$$x = \pm\, a, y = -bz,$$

and it therefore follows that these four lines will be the sections made on the surface by two tangent planes to the asymptotic cone drawn through any diametrically opposite points in its surface; these will cut each other on the surface two and two, and include an angle equal to that between the asymptotes, so that the surface may be generated by the revolution of either of these intersecting lines.

We shall shortly see that hyperboloids of one sheet, in general, admit of two distinct modes of generation by the motion of a straight line.

(147.) Eliminating the indeterminate function φ, which depends on

the nature of the generating curve (1) by differentiation, as, in the preceding problems, we find

$$\frac{p'+a}{q'+b}=\frac{x-\alpha+p'z}{y-\beta+q'z},$$

from which results the partial differential equation

$$(y-\beta-bz)\,p'-(x-\alpha-az)\,q'-b\,(x-\alpha)-a\,(y-\beta)=0\ .\ .\ (1),$$

and when the axis of z coincides with that of revolution, this becomes

$$yp'-xq'=0.$$

The differential equation of surfaces of revolution may also be obtained from the consideration of the normal, which must always cut the axis of revolution, being situated in the meridian plane. Thus the equations to the normal are (125)

$$\left.\begin{array}{l} x-x'+p'\,(z-z')=0 \\ y-y'+q'\,(z-z')=0 \end{array}\right\}$$

and as these must exist simultaneously with the equations (2), we may eliminate x, y, z, and the result will necessarily be the required relation between p', q', and the variable coordinates x', y', z', of any point on the surface.

PROBLEM VI.

(148.) A given curve surface revolves round a given axis, to determine the surface which touches and envelopes the moveable surface in every position.

The enveloping surface touches the moveable one in every position; if, therefore, we take any particular position of the latter, their combination will give the curve of contact; this curve being common to both surfaces, the tangent planes, at all its points, are common to both surfaces; hence, the values of p', q', which vary only with the tangent plane, are the same for both surfaces, as far as this common curve is concerned, and it is evidently by the revolution of this curve round the fixed axis, that the enveloping surface is generated. Hence, to determine this curve, we must deduce p', q', from the given equation, substitute them in the general equation (1) of surfaces of revolution, since there is a line on some such surface to which they belong, as well as to the given surface; and then, to determine what this line really is, it will be necessary merely to combine this last result

with the equation of the given surface: we shall thus obtain the equations of the generating curve, and the position of the fixed axis being previously known, the enveloping surface is determinable by Prob. V.

(149.) As an illustration of this, let us suppose a spheroid to revolve about any diameter, to find the equation of the surface enveloping it in every position.

Let the surface be referred to the principal diameters of the spheroid, then the equations of any other diameter will be

$$x = az,\ y = bz \ldots . (1),$$

and the spheroid itself may be represented by the equation

$$x^2 + y^2 + n^2z^2 = m^2,$$

from which we derive

$$p' = -\frac{1}{n^2} \cdot \frac{x}{z},\ q' = -\frac{1}{n^2} \cdot \frac{y}{z},$$

substituting these values in the general equation, for all surfaces of revolution round the proposed axis (1), that is in the equation

$$(y - bz)\,p' - (x - az)\,q' + ay - bx = 0,$$

and we have

$$(ay - bx)\,(1 - \frac{1}{n^2}) = 0,$$

$$\therefore ay = bx,$$

hence, combining this with the given equation, we have, for the generating curve of the envelope, the equations

$$\left.\begin{array}{c} x^2 + y^2 + n^2z^3 = m^2 \\ ay = bx \end{array}\right\} \ldots . (2),$$

hence, the envelope itself is to be determined thus. We must eliminate x, y, z, by means of (2), and the equations

$$\left.\begin{array}{c} z + ax + by = c \\ x^2 + y^2 + z^2 = r^2 \end{array}\right\} \ldots . (3),$$

of any circular section, the result will be

$$(r^2n^2 - m^2)\,(a^2 + b^2) = (c\sqrt{n^2 - 1} - \sqrt{m^2 - r^2})^2,$$

putting for r and c their values in terms of x, y, z, we have finally,

$$\{n^3\,(x^2 + y^2 + z^2) - m^2\}\,(a + b^2),$$

$$= \{(z + ax + by)\sqrt{n^2 - 1} - \sqrt{m^2 - x^2 - y^3 - z^2}\}^2,$$

which is the equation of the enveloping surface.*

* This solution is from *Hymer's Geometry of Three Dimensions*, p. 145.

CHAPTER III.

ON THE CURVATURE OF SURFACES IN GENERAL.

(150.) The simplest method of contemplating surfaces, is by considering them as produced by the motion of a line straight or curved, which, in all its positions, is subject to a fixed law. Viewed under this aspect, surfaces seem to divide themselves into two distinct and very comprehensive classes, viz. those whose generatrices must necessarily be curves, and those whose generatrices may be a straight line. If, in this latter class of surfaces, the law which regulates the generating straight line be such that through any two of its positions, however close, a plane may always be drawn, then it is obvious, that in every such surface, if a plane through the generatrix in any position, but not through any other points of the surface, that is if a tangent plane, be drawn, this plane, if supposed perfectly flexible, might be wrapped round the surface, without being twisted or torn, or, on the contrary, the surface itself might be unrolled, and would then coincide in all its points with the plane. Surfaces of this kind are, therefore, very properly distinguished by the name *Developable Surfaces*; the simplest of these are the cone and cylinder.

(151.) We see, therefore, that these surfaces are such that a plane may be drawn through any two positions of the generatrix, and which if turned round one position supposed fixed, will pass through all the intermediate positions of the other. But if the law of generation is such that this cannot have place for any two positions, however close, then the tangent plane, through one position, could plainly never be brought to pass also through another position, however near, without being *twisted*. Such surfaces, therefore, are properly designated by the name *Twisted Surfaces*.

These two kinds of surfaces will be separately discussed hereafter, the particulars in the present chapter relate to curve surfaces in general.

Osculation of Curve Surfaces.

(152. Let the equations of two curve surfaces be

$$z = f(x, y), \; Z = F(x, y),$$

when referred to the same axes of coordinates. The first of these surfaces we shall suppose fixed, both in magnitude and position by the constants a, b, c, &c., which enter its equation, being fixed. The second surface we shall suppose fixed only in form, by the form of its equation being given, but indeterminate as to magnitude and position, on account of the arbitrary constants, A, B, C, &c., which enter its equation.

Let now the variables x, and y, take the increments h and k, then, for the first surface, we have (60)

$$z' = z + \frac{dz}{dx} h + \frac{dz}{dy} k + \tfrac{1}{2} \left(\frac{d^2z}{dx^2} h^2 + 2 \frac{d^2z}{dx\,dy} hk + \frac{d^2z}{dy^2} k^2\right) + \&c.$$

and for the second,

$$Z' = Z + \frac{dZ}{dx} h + \frac{dZ}{dy} k + \tfrac{1}{2} \left(\frac{d^2Z}{dx^2} + 2 \frac{d^2Z}{dx\,dy} hk + \frac{d^2Z}{dy^2} k^2\right) + \&c.$$

or, more briefly,

$$z' = z + p'h + q'k + \tfrac{1}{2} (r'h^2 + 2s'hk + t'k^2) + \&c.$$
$$Z = Z + P'h + Q'k + \tfrac{1}{2} (R'h^2 + 2S'hk + T'k^2) + \&c.$$

Now the constants A, B, C, &c. being arbitrary, we may determine one of them in functions of x, y and the known constants, so that the condition

$$z = Z \text{ or } f(x, y) = F(x, y)$$

may be fulfilled. Such a value substituted for the constant in the equation $Z = F(x, y,)$ will cause all the surfaces represented by this equation to have a point $(x, y, z,)$ in common with the given surface. If two more of the arbitrary constants be determined from the conditions

$$p' = P', \; q' = Q',$$

the resulting values of these constants being also substituted in the same equation, the surfaces then represented will, in consequence, all have a common tangent plane at the point $(x, y, z,)$ with the fixed

surface. Therefore, that this may be the case, three arbitrary constants, at least, must enter the proposed equation, and the contact which they determine is called *contact of the first order*. *Contact of the second order* requires that the following additional conditions be fulfilled, viz.

$$r' = R', s' = S', t' = T',$$

requiring three more arbitrary constants to be determined, and so on; and that surface, all whose arbitrary constants are determined agreeably to these conditions, will, for reasons similar to those assigned at (87) for plane curves, touch the proposed surface more intimately than any other surface of the same order. It is called the *osculating surface* of that order.

If the touching surface be a sphere, then, since in its equation there can enter only *four* disposable constants, the contact cannot be so high as the second order, seeing that for this there must be *six* disposable constants, but as contact of the first order would leave still one constant arbitrary, it follows that an infinite number of spheres may have simple contact with a surface at any proposed point, yet one of these *may* be determined that shall be strictly the osculating sphere, or which shall touch more intimately all round the point of contact than any other.

Curvature of different Sections.

PROBLEM I.

(153.) At any point on a curve surface to find the radius of curvature of a normal section.

For greater simplicity, let us suppose the plane of xy to coincide with the tangent plane at the proposed point, then the axis of z will coincide with the normal, and all the normal sections will be vertical. Let the plane of the proposed section be inclined at an angle θ to the plane of xz, then the angle which its trace x' on the plane of xy makes with the axis of x will obviously be θ, and the x, y of this trace will also be the x, y of the section. Now (97) the radius of curvature ρ at the proposed point where, (86), $(dx') = (ds)$ and $\frac{(dz)}{(dx')} = 0$, is

$$\rho = \frac{\dfrac{ds^2}{dx^2}}{\dfrac{d^2z}{dx^2}} = \frac{\dfrac{ds^2}{dx^2}}{r' + 2s'\dfrac{dy}{dx} + t'\dfrac{dy^2}{dx^2}}$$

But (86)

$$\frac{dx'^2}{dx^2} = \frac{ds^2}{dx^2} = 1 + \frac{dy^2}{dx^2} = 1 + \tan.^2\theta;$$

hence, by substitution,

$$\rho = \frac{1 + \tan.^2\theta}{r' + 2s\tan.\theta + t'\tan.^2\theta} = \frac{1}{r'\cos.^2\theta + 2s\cos.\theta\sin.\theta + t'\sin.^2\theta} \ldots (1).$$

For the radius of curvature ρ', of a second normal section inclined at an angle $\theta + 90°$ to the plane of xz, we have, by putting $\theta + 90°$ for θ,

$$\rho' = \frac{1}{r'\sin.^2\theta - 2s'\cos.\theta\sin.\theta + t'\cos.^2\theta} \ldots . (2),$$

consequently,

$$\frac{1}{\rho} + \frac{1}{\rho'} = r' + t' \ldots . (3),$$

so that *the sum of the curvatures of any two normal sections through the same point at right angles to each other, is a constant quantity.*

(154.) Consequently, when one of these curvatures is the *greatest possible*, the other must be the *least possible;* that is, at every point on a curve surface, *the sections of greatest and least curvature are always perpendicular to each other*, which beautiful theorem was first discovered by *Euler*, and is demonstrated by most writers on curve surfaces, though in a manner far less simple than that above.

(155.) To determine the values of the radii of curvature of any perpendicular sections at their point of intersection, let the plane of xz be made to coincide with one of them by turning round the normal; that is to say, let $\theta = 0$, then the foregoing expressions for ρ and ρ' become

$$\rho = \frac{1}{r'}, \rho' = \frac{1}{t'} \ldots . (4).$$

(156.) But to determine the expressions for the radii of *greatest* and *least* curvatures, without causing the vertical planes of coordinates to coincide with the sections, we must know the inclinations of these

sections to the vertical planes, that is, we must know the angle θ. To find this from the property $\frac{1}{\rho} = max.$ or $min.$ we have, taking θ for the independent variable in the expression for ρ,

$$\frac{d\frac{1}{\rho}}{d\theta} = -2r' \cos.\theta \sin.\theta + 2s' (\cos.^2\theta - \sin.^2\theta) +$$
$$2t' \sin.\theta \cos.\theta = 0 \ldots. (5),$$

or, dividing by $2 \sin.^2\theta$, we have

$$\cot.^2\theta + \frac{t' - r'}{s'} \cot.\theta - 1 = 0 \ldots. (6),$$

from which we get for the two inclinations sought

$$\cot.\theta = \frac{r' - t' \pm \sqrt{(r' - t')^2 + 4s'^2}}{2s'} = \frac{\cos.\theta}{\sin.\theta},$$

the upper sign corresponding to the maximum, and the under to the minimum. Substituting these values in the first of the expressions (1), which may be written thus

$$\rho = \frac{\cot.^2\theta + 1}{r' \cot.^2\theta + 2s' \cot.\theta + t'},$$

we have for the radii of greatest and least curvatures the expressions

$$\left.\begin{aligned} r &= \frac{2}{r' + t' + \sqrt{(r' - t')^2 + 4s'^2}} \\ R &= \frac{2}{r' + t' - \sqrt{(r' - t')^2 + 4s'^2}} \end{aligned}\right\} \ldots (7).$$

These are called the *principal radii of curvature* at the proposed point, and the sections themselves the *principal sections* through that point.

(157.) If we know the principal radii and the inclination φ of any normal section to a principal section through the point, the radius of curvature of the normal section at that point may be expressed in terms of these known quantities. For, bringing the vertical coordinate planes into coincidence with the planes of principal section, we have $\theta = 0$, and, consequently, as appears from equation (6), last article, $s' = 0$; and, since (4)

$$\frac{1}{r} = r', \frac{1}{R} = t',$$

we have

$$\rho = \frac{1}{r' \cos.^2 \varphi + t' \sin.^2 \varphi} = \frac{Rr}{r \sin.^2 \varphi + R \cos.^2 \varphi} \ldots (8)$$

$$\therefore \frac{1}{\rho} = \frac{1}{R \sin.^2 \varphi} + \frac{1}{r \cos.^2 \varphi} \ldots (9).$$

It is plain from this expression that if R and r have the same sign, ρ will have that sign for every section through the proposed point, which is the same as saying that if the principal sections are both convex or both concave, every other section through the same point will be similarly convex or concave, and, therefore, also the entire surface at that point. In such a case the minimum radius must be absolutely shorter than any other radius of curvature at the point, and the maximum radius longer than any other.

(158.) If the two principal radii have not only the same sign but the same length, then the foregoing expression gives always $\rho = R$ whatever be the inclination φ, so that then all the normal sections have the same curvature and all are *principal sections*, as is the case with the sphere and with the ellipsoid of revolution, the paraboloid of revolution, &c. at those points through which the fixed axis passes.

(159.) If the surface belong to the second of the classes mentioned in (147), then no point can be assumed on it through which a straight line may not be drawn, and, as the curvature of this line is 0, it follows that the curvature of the section perpendicular to it must be equal to the sum of the curvatures of any two perpendicular sections through the same point.

(160.) Let us now suppose that the principal radii R, r have different signs, as r positive and R negative, which will be the case if one of these sections be convex and the other concave, we shall then have

$$\rho = \frac{Rr}{R \cos.^2 \varphi - r \sin.^2 \varphi},$$

which becomes infinite when

$$r \sin.^2 \varphi = R \cos.^2 \varphi \text{ or when } \tan. \varphi = \pm \sqrt{\frac{R}{r}},$$

but for all positive and negative values of φ between this and 0, ρ will be positive, while beyond these limits ρ will be negative.

It appears, therefore, that if from the origin two straight lines be

drawn in the tangent plane inclined to the axis of x at the angles $\varphi = + \sqrt{\frac{R}{r}}$ and $\varphi = - \sqrt{\frac{R}{r}}$, these will coincide with the surface; all the sections between the sides of the two opposite angles thus formed will be convex, all the sections between the sides of the other two opposite supplementary angles will be concave, so that the two straight lines which we have seen may be drawn from the proposed point to coincide with the surface, separate the convexity from the concavity at that point.

(161.) In order to determine whether the principal radii at any point are both of the same sign or not, we may observe that the expressions (7) for these radii at art. (156) may be put under the form

$$\left.\begin{aligned} r &= \frac{2}{r' + t' + \sqrt{(r' + t')^2 - 4(r't' - s'^2)}} \\ R &= \frac{2}{r' + t' - \sqrt{(r' + t')^2 - 4(r't' - s'^2)}} \end{aligned}\right\} \ldots . (10),$$

from which forms we immediately see that the radii will have the same sign, viz. positive if $r't' - s'^2 > 0$, and contrary signs if $r't' - s'^2 < 0$; this last condition, therefore, exists in the case just considered.

(162.) We shall terminate these remarks by showing that a paraboloid of the second order may always be found, such that its vertex being applied to any point in any curve surface, the normal sections through that point shall have the same curvature for both surfaces.

For, take the planes of the principal sections for those of xz, yz, then the radii of these sections being R, r we know that a paraboloid, whose vertex is at the origin, will in reference to the same axes be represented by the equation (*Anal. Geom.*)

$$z = \frac{x^2}{2r} \pm \frac{y^2}{2R},$$

r and R being the semi-parameters of the sections of the paraboloid on the planes of xz, yz. Now the equation of a normal section of this paraboloid, by a plane whose inclination to that of xz is φ, will be obtained by substituting in this equation $x' \cos.\varphi$ for x, $x' \sin.\varphi$ for y, z remaining the same for all normal sections (*Anal. Geom.*); hence, the equation of the section in question is

$$z = \left(\frac{\cos.^2\varphi}{2r} \pm \frac{\sin.^2\varphi}{2R}\right) x'^2 \; \therefore \; x'^2 = \frac{2Rr}{R\cos.^2\varphi \pm r\sin.^2\varphi} z.$$

so that the semi-parameter, and, consequently, the radius of curvature (94) of this parabolic or hyperbolic section, is

$$\frac{\mathrm{R}r}{\mathrm{R}\cos.^2\varphi \pm r\sin.^2\varphi},$$

the very same as the radius of curvature of the corresponding section of the proposed surface, be this what it may (154). Hence, this paraboloid has the same curvature in every direction that the proposed surface has at the origin of the coordinates.

PROBLEM II.

(163.) To determine the radius of curvature at any point in an oblique section.

Take the tangent to the section through the point as axis of x, the point itself for the origin, and the axis of z' in the plane of the section; then, calling the normal the axis of z, the normal section through the axis of x, s, and the oblique section s', we have, at the proposed point (86), $(ds) = (ds')$. Now at the proposed point $\gamma = \frac{(ds')^2}{(d^2z')}$, but if the axis of z' be transferred to the axis of z, then

$$z = z'\cos.\theta \therefore (d^2z) = (d^2z')\cos.\theta;$$

hence, by substitution,

$$\gamma = \frac{(ds)^2}{(d^2z)}\cos.\theta = \rho\cos.\theta \quad . \; . \; . \; . \quad (1),$$

where γ is the radius of the oblique section, and ρ the radius of the normal section through the tangent to the former; so that *γ is the projection of ρ on the plane of the oblique section*, which remarkable property is the *theorem of Meusnier*.

It immediately follows from this theorem, that, if with the radius of any normal section of a curve surface a sphere be described, and through the tangent to that section at the normal point planes be drawn, cutting both the sphere and the proposed surface, every section of the sphere will be an osculating circle to the corresponding section of the surface, because, if the normal radius of the sphere be projected on any of these sections, the projection will obviously be the radius of that section, and the same projection is, by the above theorem, the radius of curvature of the corresponding section of the proposed surface.

Lines of Curvature and Radii of Spherical Curvature.

(164.) In speaking of plane curves we have already explained (104) what is to be understood by *consecutive normals* and *consecutive curves.* We propose, in the present article, to consider the intersections of any normal at a point of a curve surface with its consecutive normal; but here it must be remarked that consecutive normals to curve surfaces do not necessarily intersect, as in plane curves, for, before coinciding, these normals, although ever so close, need not be both in the same plane; and, in such a case, when they become consecutive, or coincide, they coincide throughout at once, having even then no point in common that before coinciding was a point of intersection. Hence such consecutive normals have no point of intersection. If, however, upon any curve surface there can be traced a line, such that the normal to the surface at every point of it is intersected by the consecutive normal, that line will have peculiar properties. Such a line is called, by *Monge, a line of curvature.*

PROBLEM III.

(165.) To determine the lines of curvature through any point on a curve surface.

Let the surface be referred to any rectangular axes whatever, then $(x', y', z',)$ being any point on it, we have, for the equations of the normal,

$$\left.\begin{array}{ll}(\mathrm{A}) & x - x' + p'(z - z') = 0 \\ (\mathrm{B}) & y - y' + q'(z - z') = 0\end{array}\right\} \quad \ldots . \ (1).$$

Let now the independent variables x', y', take any increments h, k, the equations of the normal to the corresponding point will be

$$\left.\begin{array}{l}\mathrm{A} + \dfrac{d\mathrm{A}}{dx'}h + \dfrac{d\mathrm{A}}{dy'}k + \&\mathrm{c.} = 0 \\ \mathrm{B} + \dfrac{d\mathrm{B}}{dx'}h + \dfrac{d\mathrm{B}}{dy'}k + \&\mathrm{c.} = 0\end{array}\right\} \quad \ldots . \ (2).$$

Now, if the normals (1), (2) intersect, their equations must exist simultaneously; therefore, since $\mathrm{A} = 0$, $\mathrm{B} = 0$,

$$\left.\begin{array}{l}\dfrac{d\mathrm{A}}{dx'} + \dfrac{d\mathrm{A}}{dy'}\dfrac{k}{h} + \&\mathrm{c.} = 0 \\ \dfrac{d\mathrm{B}}{dx'} + \dfrac{d\mathrm{B}}{dy'}\dfrac{k}{h} + \&\mathrm{c.} = 0\end{array}\right\} \quad \ldots . \ (3).$$

The coordinates $(x, y, z,)$ of the intersection of the proposed normals will be obtained by the combination of the four equations (1) and (3) in terms of x', y' z', which are fixed, and of the increments k, h. But from four equations three unknowns may be always eliminated, and the result of this elimination will be an equation between the other quantities; hence then there exists a *constant* relation between the increments k, h, when the normals intersect, these increments are therefore *dependent;* consequently the y, x, of which these are the increments, must be dependent;* therefore when the normals are consecutive, that is, when $h = 0$, the equations (3) become

$$\left.\begin{aligned} \frac{dA}{dx'} + \frac{dA}{dy'} \cdot \frac{dy'}{dx'} = 0 \\ \frac{dB}{dx'} + \frac{dB}{dy'} \cdot \frac{dy'}{dx'} = 0 \end{aligned}\right\} \dots \; (3'),$$

or, by substituting for A and B their values (1),

$$1 + p'\left(p' + q'\frac{dy'}{dx'}\right) + (z' - z)\left(r' + s'\frac{dy'}{dx'}\right) = 0 \dots (4),$$

$$\frac{dy}{dx'} + q'\left(p' + q'\frac{dy'}{dx'}\right) + (z' - z)\left(s' + t'\frac{dy'}{dx'}\right) = 0 \dots (5),$$

from which, eliminating $z' - z$, we have the following equation for determining $\frac{dy'}{dx'}$

$$((1 + q'^2)\,s' - p'q't')\frac{dy'^2}{dx'^2} + ((1 + q'^2)\,r' - (1 + p'^2)t')\frac{dy'}{dx'} -$$
$$(1 + p'^2)\,s' + p'q'r' = 0 \quad \dots (6).$$

This being a quadratic equation furnishes two values for $\frac{dy}{dx}$ the tangent of the inclination of the projection of the line of curvature through (x', y', z'), on the plane of xy to the axis of x. Hence, there are two directions in which lines of curvature can be drawn through any proposed point, and if in (6) we substitute for p', q', &c. their general values in functions of x, y, that equation will then be the differential equation which belongs to the projections of *every* pair of

* If this should appear doubtful to the student, its truth may be shown by removing the axes of x, y, to the proposed point, in which position k, h, will be the variable coordinates of the line of curvature, and these will merely take a constant when the axes are replaced in their first position.

lines of curvature; so that every line on a curve surface which at all its points satisfies this equation, will be a line of curvature.

(166.) Between every pair of lines of curvature there exists a very remarkable relation: it is that they are always *at right angles to each other.* To prove this it will only be necessary to place the coordinate planes, which have hitherto been arbitrary, so that the plane of xy may coincide with, or at least be parallel to, the tangent plane at the point to be considered, in which case p' and q' are both 0, and, consequently, the equation (6) becomes

$$\frac{dy^2}{dx^2} + \frac{r' - t'}{s'} \cdot \frac{dy}{dx} - 1 = 0 \quad . \; . \; . \; . \quad (7),$$

therefore, calling the two roots or values of $\frac{dy}{dx}$, tan. φ and tan. φ', we have, by the theory of equations,

$$\tan.\,\theta \tan.\,\theta' = -1,\text{*}$$

which proves that the projections of the two lines of curvature through the origin, are perpendicular to each other, and consequently the lines themselves are perpendicular to each other.

Moreover, the equation (7), if divided by $\frac{dy^2}{dx^2} = \tan.^2\,\theta$ becomes identical to equation (6), page 183, which determines the inclinations of the principal sections; hence, the lines of curvature through any point, always touch the sections of greatest and least curvature at that point. Also, in the same hypothesis, with respect to the disposition of the coordinate planes $z' = 0$, therefore the equation (4) or (5) gives

$$z = \frac{1}{r' + s \tan.\,\theta} \text{ or } = \frac{\tan.\,\theta}{s' + t' \tan.\,\theta},$$

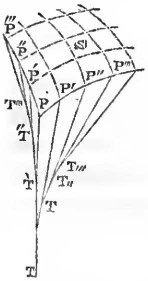
but if the plane of xz coincide with a plane of principal section, it will, as we have just seen, touch the line of curvature, and then $\theta = 0$, so that

$$z = \frac{1}{r'} \text{ or } \frac{1}{t'},$$

and these are precisely the expressions found at (152), for the two radii of curvature of the principal sections at the proposed point, in

* Since tangent ϕ and tangent ϕ' are the roots of equation, (7), and -1 is their product, recollecting that tangent $\times$ cot. $=$ radius$^2 = 1$, whence ϕ' is the complement of ϕ. Ed.

reference to the same axes; hence we infer, (161), that the consecutive normals to the surface at any point, intersect at the same points as the consecutive normals to the principal sections. These points of intersection, are no other than the centres of curvature of the surface at the proposed point, for if spheres be described from these centres to pass through the proposed point, they will touch there, since both have the same normal, and therefore the same tangent plane; and these two spheres have the same curvature as the surface in the two directions of the lines of curvature, since consecutive normals to the surface in these directions, cut that through the point at the centres of these spheres, also the plane sections, tangential to these directions, have the corresponding sections of the spheres for their osculating circles, since the consecutive normals, at their point of contact, also intersect at these centres; therefore, the radii of curvature of the surface at any point, coincides entirely with the radii of curvature of the principal sections through that point, so that (155) if the radii are both equal at any point, the curvature of the surface is uniform all round that point.



(167.) The annexed figure is intended to give an idea of the disposition of the lines of curvature on the surface (S), drawn through points P, P′, &c. PT, P′T′, &c. are the normals to the surface at those points, and as each is intersected by its consecutive normal, the locus TT′ . . . of these intersections is a curve. The locus too of the normals PT, P′T′, &c. themselves form a surface, throughout perpendicular to the proposed; this surface, thus generated by the motion of a straight line PT along the curve PP′ . . . and each position intersecting its consecutive position, is obviously a *developable surface;* one of whose edges is the line of curvature PP′ . . . and the other the line of centres TT′ . . . which latter is called the *edge of regression of the developable surface.* Proceeding, in like manner, along the other line of curvature through P, we have another developable normal surface, whose edge of regression is the locus of the centres of curvature belonging to this second line of curvature. Applying similar considerations to *every* point on the surface (S), we shall thus have an infinite number of developable normal surfaces at right angles to

each other, and which will obviously form together two continuous volumes, and the edges of regression will, in like manner form two continuous surfaces, or *sheets*, being the locus of all the centres of curvature. These surfaces, therefore, bear the same relation to the original surface, as that which in plane curves we have called the evolute bears to the involute.

It would be quite incompatible with the pretensions of this little volume to extend any further our inquiries into the properties of *lines of curvature.* For more detailed information respecting these remarkable lines, the student must study the illustrious author by whom they were first considered, Monge, in his *Application de l'Analyse à la Géométrie*, a work abounding with the most profound and beautiful speculations on the subject of curve surfaces and curves of double curvature, and which, together with the *Developpements de Géométrie* of *Dupin*, constitute a complete body of information on a very attractive and important branch of mathematical study, the cultivation of which, however, has been almost entirely neglected hitherto in this country.*

Radii of Spherical Curvature.

(168.) We have already seen that the radii of spherical curvature, or simply the radii of curvature at any point of a surface, are identical to the radii of the principal sections through that point, and have given tolerably commodious formulas for the calculation of these radii when the axes to which the surface is referred originate at the proposed point, the plane of xy being coincident with the tangent plane, and the axis of z with the normal at that point. We have also seen that when these radii are determined, a paraboloid may also be determined, having its vertex at the proposed point and its curvature in all directions round that point and in its immediate vicinity, the same as the curvature of the surface; so that be the surface ever so complicated, its curvature at any particular point will be correctly presented to us by the vertex of a determinable paraboloid. All this, however, supposes the radii of curvature of the surface at this point to be known;

* The only English Mathematician, I believe, who has produced public proof of his having given much attention to these inquiries, is *Mr. Davies* of Bath, whose papers on surfaces, &c. in *Leybourn's Repository*, I have already had occasion to refer to.

it remains, therefore, to show how these radii may be determined, whatever be the position of the coordinate axes.

PROBLEM IV.

(169.) Given the coordinates of a point on a curve surface to determine the radii of curvature at that point.

Let $(x, y, z,)$ be the point on the surface, and $(x', y', z',)$ either of the sought points on the normal corresponding to the centres of curvature, then the radius R from either will be given by the expression

$$R^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2.$$

Since at the proposed point the equations (1) and (3), at art. (162), must exist simultaneously with this, we have, by substituting in this the values of $(x' - x)$, $(y' - y)$ as given by (1),

$$R = (z' - z) \sqrt{1 + p'^2 + q'^2}.$$

Now, if from (4), (5) we eliminate the unknown $\frac{dy}{dx}$, we have

$$(z - z')^2 (r' t' - s'^2) + (z - z') \{(1 + q'^2) r' - 2p' q' s' + (1 + p'^2) t'\} + (1 + p'^2 + q'^2) = 0,$$

or, putting according to Monge

$$g = r' t' - s'^2$$
$$h = (1 + q') r' - 2p' q' s' + (1 + p'^2) t'$$
$$k^2 = 1 + p'^2 + q'^2$$

the equation for determining $z - z'$ becomes

$$(z - z')^2 + \frac{h}{g} (z - z') + \frac{k^2}{g} = 0 \quad . \; . \; . \; . \quad (1),$$

and the roots of this substituted in the equation

$$R = (z - z') k,$$

give

$$R = \frac{k}{2g} (h \pm \sqrt{h^2 - 4gk^2}) \quad . \; . \; . \; . \quad (2)$$

$$= \frac{2k^3}{h \pm \sqrt{h^2 - 4gk^2}} \quad . \; . \; . \; . \quad (3).$$

(170.) Thus the radii of curvature are determined, and the directions of the lines of curvature, and therefore also of the principal sec-

tions are determined by Problem III.; consequently, the radius of curvature of an oblique section, any how inclined to coordinate planes, any how situated with respect to the surface, may now be determined by help of the formulas (9) and (1) at pages 181 and 184. It appears from (3) that the surface will be convex or concave in the direction of a line of curvature in the immediate vicinity of the point, according as $g \gt 0$ or $g \lt 0$. If $g = 0$ the equation (2) shows that one of the radii will be infinite.

When the functions of x, y, z, represented by p', q', r', s', t', are complicated, the expressions just deduced for the radii of curvature will obviously be complicated in the extreme. They are, however, easily manageable when the proposed surface is of the second order, as *Dupin* has shown in his *Developpements* for both classes of these surfaces. We shall here give the solution for surfaces which have not a centre, that is for paraboloids; the process for the other class, or for central surfaces, being exactly the same but rather longer.

PROBLEM V.

(171.) To determine the radii of curvature at any point in a paraboloid.

The general equation of paraboloids being

$$\frac{x^2}{A^2} = \frac{y^2}{B^2} + 2z = 0,$$

we have

$$p' = -\frac{x}{A}, q' = -\frac{y}{B} \therefore 1 + p'^2 + q'^2 = \frac{x^2}{A^2} + \frac{y^2}{B^2} + 1 = k^2$$

$$r' = -\frac{1}{A}, s' = 0, t' = -\frac{1}{B} \therefore r't' - s'^2 = \frac{1}{AB} = g$$

$$\therefore h = -(1 + \frac{x^2}{A^2})\frac{1}{B} - (1 + \frac{x^2}{B^2})\frac{1}{A} = -\frac{A + B - 2z}{AB}.$$

Hence, generally, whatever be the paraboloid, we have, for the coefficients in equation (1) above, the values

$$\frac{h}{g} = -A + B - 2z, \frac{k^2}{g} = AB\,(\frac{x^2}{A^2} + \frac{y^2}{B^2} + 1)^2$$

and for R we have

$$R = \sqrt{\frac{x^2}{A^2} + \frac{y^2}{B^2} + 1} \times$$

$$\left\{\frac{A + B\,2z}{2} \pm \sqrt{\left(\frac{A + B\,2z}{2}\right)^2 - AB\left(\frac{x^2}{A^2} + \frac{y^2}{B^2} + 1\right)}\right\}.$$

The sum of the two radii are, therefore,

$$R + r = \sqrt{\frac{x^2}{A^2} + \frac{y^2}{B^2} + 1} \times (A + B - 2z)$$

but (124) the first of these factors is the reciprocal of the cosine of the inclination α of the normal at the point $(x, y, z,)$ to the axis of z,

$$\therefore (R + r) \cos. \alpha = A + B - 2z,$$

which is the expression for the sum of the projections of the radii of curvature on the axis of z; A, B being the semi-parameters of the sections on the planes of xy, yz. If the point be at the vertex, then $x = 0$, $y = 0$, $z = 0$, and the values of R then become

$$R = \frac{A + B}{2} \pm \sqrt{\left(\frac{A + B}{2}\right)^2 - AB} = \frac{A + B}{2} \pm \frac{A - B}{2}$$

$$\therefore R = A,\ r = B,$$

and these are also the radii of curvature of the two parabolic sections on the planes of xy, yz (94), so that these sections which we have already called the principal sections in the Analytical Geometry, are really the principal sections, or those of greatest and least curvature. A similar process leads to similar inferences for central surfaces of the second order.

CHAPTER IV.

ON TWISTED SURFACES.*

(172). We have already stated (148) a twisted surface to be one whose generatrix is a straight line moving in such a manner along its directrices that it continually changes the plane of its motion.

* This is the class of surfaces called by the French *Surfaces Gauches*, and which, together with the class of developable surfaces, they include under the

The present chapter will be devoted to the consideration of this class of surfaces. Proceeding from the simpler kinds to the more general, we shall first examine the surfaces whose directrices are straight lines as well as the generatrices, then those having one of its directrices a curve, afterwards those having two curvilinear directrices, and lastly those having three directrices of any kind.

Twisted Surfaces having Rectilinear Directrices only.

PROBLEM I.

(173.) To determine the surfaces generated by a straight line moving parallel to a fixed plane, and along two rectilinear directrices not situated in one plane.

Let the fixed plane, called the directing plane, be taken for that of xy, and the plane parallel to the two directrices for that of xz; then the equations of these directrices will be

$$(1) \; . \; . \; . \; . \left.\begin{array}{l} x = az + \alpha \\ y = \beta \end{array}\right\} \text{ and } \left\{\begin{array}{l} x = \alpha' z + \alpha' \\ y = \beta' \end{array}\right. \; . \; . \; . \; . \; (2),$$

and the generatrix being parallel to the plane of xy will be represented by the equations

$$z = b, y = mx + n \; . \; . \; . \; . \; (3).$$

As this line has always a point in common with (1), the four equations (1), (3) exist together, therefore, eliminating x, y, z, we have, among the variable parameters, the relation

$$\beta = m\,(ab + \alpha) + n \; . \; . \; . \; . \; (4),$$

the parameters a, α, β, being fixed by the position of the directrices, but the others variable.

In like manner, since the line (3) has also always a point in common with (2), the four equations (2), (3) exist together, therefore, eliminating x, y, z, we get for a second relation among the three arbitrary parameters the equation

general name of *Surfaces Réglées*, expressive of their mode of generation by straight line generatrices. There has just appeared, in *Leybourn's Repository*, No. 22, a very masterly inquiry into the history of these surfaces, from the pen of Mr. *Davies*, wherein the claims of the English to the first consideration of "rule surfaces" is fully established.

$$\beta' = m(a'b + \alpha') + n \quad . \quad . \quad . \quad . \quad (5).$$

By means of the two relations (4) and (5) among the parameters which enter (3), we may eliminate them and thus obtain the sought equation in x, y, z. Subtracting each from (3), we have

$$y - \beta = m(x - az - \alpha)$$
$$y - \beta' = m(x - a'z - \alpha'),$$

eliminating m we obtain, finally,

$$(a - a')yz + (\alpha - \alpha')y + (a'\beta - a\beta)z + (\beta' - \beta)x$$
$$= \alpha\beta' - \alpha'\beta \quad . \quad . \quad . \quad . \quad (6)$$

for the equation of the surface, which is therefore of the second order. Let us now inquire what particular kind of surfaces of the second order this equation includes. By applying the criteria (3) (*Anal. Geom.*) we find that the surfaces are not central, they must, therefore, be paraboloids. By putting $x = k$ we find in the resulting equation for any section parallel to the plane of yz, that the squares of the variables are absent, therefore, (*Anal Geom.*) these sections are all hyperbolas. We infer, therefore, that the surface (6) is always a *hyperbolic paraboloid.* If, in the equation (6) we make z equal to any constant quantity, the equation will always be that of a straight line, being indeed necessarily one of the positions of the generatrix; also, if we put y equal to any constant quantity, we find that every section parallel to the plane of xz is a straight line, so that through every point on the surface of a hyperbolic paraboloid there may be drawn two straight lines, their assemblage constituting two distinct series situated in two distinct series of parallel planes, and hence there are two distinct ways in which the surface may be generated by the motion of a straight line, but not more than two ways, since the equation (6) represents a straight line only on the two hypotheses assumed above; and as no two of the positions of the same generatrix, however close, can be in the same plane, the hyperbolic paraboloid is a *twisted surface.*

(174.) We may show at once by setting out with the equation of the hyperbolic paraboloid, that two straight lines pass through every point on its surface, and, moreover, that these lines are both in the tangent plane at that point. Thus the equation of the surface is (*Anal. Geom.*)

$$px^2 - p'y^2 = pp'z \quad . \quad . \quad . \quad . \quad (1),$$

and that of the tangent plane through $(x', y', z',)$

$$2pxx' - 2p'yy' = pp' (z + z') \quad . \; . \; . \; . \quad (2),$$

the relation among the coordinates x', y', z', of the point of contact being of course

$$px'^2 - p'y'^2 = pp'z' \quad . \; . \; . \; . \quad (3).$$

Adding together equations (1) and (3) and subtracting (2) from the sum, there results

$$p (x - x')^2 - p' (y - y')^2 = 0,$$

which is the condition necessary to be satisfied for every projected point $(x, y,)$ common to the surface (1) and the plane (2), seeing that it has resulted from the combination of their equations. Such condition being satisfied by every point in the lines represented by the equation

$$y - y' = \pm (x - x') \sqrt{\frac{p}{p'}},$$

it follows that the lines of which these are the projections are common to both surface and tangent plane, so that the tangent plane cuts the surface according to two straight lines passing through the point of contact.

PROBLEM II.

(175.) To determine the surface generated by the motion of a straight line along three others fixed in position, so that no two of them are in the same plane.

Let us first consider the case in which the three directrices are all parallel to the same plane.

Assume the axes of x and y in this plane passing through one of the directrices (B), and parallel to the other two (B′), (B″). Let the axis of x coincide with (B), and the axis of y be parallel to (B′), and let the axis of z be drawn to pass through both (B′) and (B″), then the equations of the directrices will be

(B)	$y = 0, \; z = 0$
(B′)	$x = 0, \; z = h$
(B″)	$y = ax, z = k$

and the equation of the generatrix in any position will be

$$x = mz + p, y = nz + q \quad . \; . \; . \; . \quad (1).$$

As this line has always a point in common with the directrices, all these equations exist together. Hence, eliminating x, y, z, we have, among the variable parameters m, n, p, q, the relations

$$q = 0,\ mh + p = 0,\ nk = a\,(mk + p) \quad \ldots . \quad (2).$$

Eliminating the variable parameters from (1) and (2), we have

$$a\,(k - h)\,xz = ky\,(z - h)$$

for the equation of the surface sought, and which we find, by applying the same tests as in last problem, to be the same surface, viz. the hyperbolic paraboloid.

(176.) Suppose, now, that the three directrices are not all parallel to the same plane, then, taking any point in space for the origin, and parallels to the directrices for axes, the equations of these will be

$$\begin{array}{ll} (B) & x = \alpha,\ \ y = \beta \\ (B') & z = \gamma,\ \ x = \alpha \\ (B'') & y = \beta',\ z = \gamma' \end{array}$$

and the equation of the generatrix will be

$$x = mz + p,\ y = nz + q \quad \ldots . \quad (1),$$

which, since it has a point in common with (B), gives rise to the condition

$$\frac{\alpha - p}{m} = \frac{\beta - q}{n} \quad \ldots . \quad (2),$$

and having, at the same time, a point in common with (B), and another in common with (B''), we have the additional conditions

$$\alpha' = m\gamma + p,\ \beta' = n\gamma + q \quad \ldots . \quad (3).$$

Eliminating now the arbitrary parameters, m, n, p, q, by means of (1), and these equations of condition, we shall arrive at the equation of the surface. The equations (3) give, in conjunction with (1),

$$m = \frac{x - \alpha'}{z - \gamma},\ n = \frac{y - \beta'}{z - \gamma}$$

$$p = \alpha' - \gamma\frac{x - \alpha'}{z - \gamma},\ q = \beta' - \beta'\frac{y - \beta'}{z - \gamma'},$$

which values, substituted in (2), give

$$\left.\begin{array}{c} (\gamma - \gamma')\,xy + (\beta' - \beta)\,xz + (\alpha - \alpha')yz \\ + (\beta\gamma' - \beta'\gamma)\,x + (\alpha'\gamma' - \alpha\gamma)\,y + (\alpha'\beta - \alpha\beta')\,z \\ + \alpha\beta'\gamma - \alpha'\beta\gamma' \end{array}\right\} = 0$$

for the equation of the surface. By applying the usual criteria, (*Anal. Geom.*) we find that the surface must be a hyperboloid, and as the squares of the variables are all absent from the equation, no intersection (*Anal. Geom.*) can possibly be an imaginary curve; hence the surface must be a *hyperboloid of a single sheet*, and it is obviously twisted, since the generatrix constantly changes the plane of its motion.

(177.) We may, as in the preceding problem, by commencing with the equation of this surface, show that through every point on it two straight lines may be drawn, and that they will both be in the tangent plane through the point. Thus the equation of the surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \ldots . \quad (1),$$

and that of the tangent plane through (x', y', z')

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1 \quad \ldots . \quad (2),$$

the relation among x', y', z', being fixed by the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - \frac{z'^2}{c^2} = 1 \quad \ldots . \quad (3).$$

Adding together equations (1) and (3), and subtracting twice equation (2) from the result, we have

$$\frac{(x-x')^2}{a^2} + \frac{(y-y')^2}{b^2} - \frac{(z-z')^2}{c^2} = 1 \quad \ldots . \quad (4),$$

a relation which must have place for every point common to both the surface and the tangent plane.

Also, subtracting (3) from (2)

$$\frac{x'(x-x')}{a^2} + \frac{y'(y-y')}{b^2} - \frac{z'(z-z')}{c^3} = 0 \quad \ldots . \quad (5).$$

Now, in order to ascertain whether the points fulfilling these conditions can lie in a straight line, let us combine them with the equations of a straight line through $(x', y', z',)$ viz.

$$x - x' = a'(z - z'),\ y - y' = b'(z - z') \quad \ldots . \quad (6).$$

Substituting in the equations (4) and (5) these expressions for $x - x'$, $y - y'$, we have,

$$(z' - z')^2 \left(\frac{a'}{a^2} + \frac{b'^2}{b^2} - \frac{1}{c^2}\right) = 0$$

$$(z - z') \left(\frac{a'x'}{a^2} + \frac{b'y'}{b^2} - \frac{z'}{c^2}\right) = 0$$

$$\therefore \frac{a'}{a^2} + \frac{b'^2}{b^2} - \frac{1}{c^2} = 0$$

$$\frac{a'x'}{a^2} + \frac{b'y'}{b^2} - \frac{z'}{c^2} = 0$$

these relations, therefore, must exist among the constants in (6), for it to be possible for that line to belong to the surface. From the second of these we readily deduce a rational value of a' which, substituted in the first, b' will be given by the solution of the quadratic, which will furnish two values, so that two lines passing through the point of contact may be drawn, that shall be common to both the surface and the tangent plane.

Twisted Surfaces having but one Curvilinear Directrix.

(178.) In surfaces of this kind the generatrix moves along a straight line and a curve, remaining constantly parallel to a fixed plane called the *directing plane.* Such surfaces are called *conoids*, and that they are twisted surfaces is plain, because a plane to pass through two positions of the generatrix must pass through the rectilinear directrix, and become, therefore, fixed, so that it cannot be moved round one position without ceasing to pass through two. The directing plane is usually taken for that of xy, the origin being at the point where the straight directrix pierces it.

PROBLEM III.

(179.) To determine the general equation of canoidal surfaces:

Let the equations of the straight directrix be

$$x = mz, \; y = nz \; . \; . \; . \; . \; (1),$$

and those of the curvilinear directrix,

$$\mathrm{F}\,(x, y, z) = 0, f(x, y, z) = 0 \; . \; . \; . \; . \; (2).$$

The equation of the generatrix, being in every position parallel to the plane of xy, must always be of the form

$$z = \alpha, y = \beta x + \gamma \;.\;.\;.\;.\; (3),$$

α and β being variable parameters.

As this line has always a point in common with the line (1), their equations exist together; hence, eliminating x, y, z, by means of these four equations, we have the condition

$$n\alpha = \beta m\alpha + \gamma \text{ or } \gamma = n\alpha - \beta m\alpha,$$

so that the equations (3) of the generatrix become

$$z = \alpha, y - n\alpha = \beta\,(x - m\alpha) \;.\;.\;.\;.\; (4);$$

but this same line has also a point in common with the curve (2); hence, eliminating x, y, z, by means of the four equations (2), (4), we have an equation containing only constants and the variable parameters α, β, which equation, solved for α, gives

$$\alpha = \varphi : \beta \;.\;.\;.\;.\; (5).*$$

But, by equations (4),

$$\alpha = z, \beta = \frac{y - nz}{x - mz};$$

hence, by substitution in (5),

$$z = \varphi\,(\frac{y - nz}{x - mz}),$$

which expresses the general relation among the coordinates of any point of the generatrix in any position, therefore this is the general equation of a conoidal surface.

(180.) If the straight directrix coincide with the axis of z, then $m = 0$, $n = 0$, and the conoid is represented by the general equation

$$z = \varphi\,(\frac{y}{x}),$$

whether the axis of z, or the straight directrix, be perpendicular to the directing plane or not; if it is perpendicular, the conoid is called a *right conoid*. In these cases the equations of the generatrix are simply

$$z = \alpha, y = \beta x.$$

(181.) As an example, let it be required to find the equation of the inferior surface of a winding staircase, the aperture or column round which it winds being cylindrical.

* $\phi : \beta$ is the same as $\phi\beta$ or a function of β. *Ed.*

To conceive the generation of this surface, let us suppose a rectangle to be rolled round a vertical column, which it just embraces, the line which was the diagonal of the rectangle will then become a winding curve called a *helix*, and it will make just one turn round the column, its horizontal projection being a circle; if immediately above this another equal rectangle be applied to the column, the vertical edges when brought together being in a line with those of the first, the diagonal of this will form a continuation of the helix, and in this way will be exhibited the trace of the edge of the surface in question on the vertical column, or the curvilinear directrix; the other directrix is the axis of the cylinder, the directing plane being horizontal.

Now for every point in the diagonal of a rectangle the abscissa has a constant ratio to the ordinate, the axes being the sides including the diagonal, so that, reckoning from the foot of the helix, the circular abscissas and vertical ordinates corresponding are in a constant ratio. Hence, taking the centre of the cylindrical base for the origin and drawing the axis of y through the foot of the helix, calling h the height and $2\pi r$ the base of one of the rectangles, or of the cylinder, we shall have for each point of the helix, these relations, viz.

$$x^2 + y^2 = r^2,\ x = r \sin. \frac{s}{r},\ \frac{z}{s} = \frac{h}{2\pi r} \ \ldots\ldots (1),$$

and for the generating line the equations

$$z = \alpha,\ y = \beta x \ \ldots\ldots (2).$$

If from the two last of (1) we eliminate the arc s we shall have the following equations of the projections of the curve

$$x + y^2 = r^2,\ x = r \sin. \left(\frac{2\pi z}{h}\right) \ldots\ldots (3),$$

eliminating x, y, z, from the equations (2), (3) we have

$$\frac{1}{\sqrt{1 + \beta^2}} = \sin. \left(2\pi \frac{\alpha}{h}\right),$$

in which equation if we substitute for α and β the values z and $\frac{y}{x}$ given by (2), we shall obtain, finally,

$$\frac{x}{\sqrt{x^2 + y^2}} = \sin. \left(2\pi \frac{z}{h}\right) \text{ or } \frac{x}{y} = \tan. \left(2\pi \frac{z}{h}\right)$$

which is the equation of the surface, that is of the *twisted helixoid*.

(182.) It remains to determine the differential equation of conoidal surfaces. In order to this we must eliminate the arbitrary function φ in the equation

$$z = \varphi\left(\frac{y - nz}{x - mz}\right),$$

by differentiating, as in the several similar cases in Chapter II., we thus obtain the equation

$$\frac{p'}{q'} = \frac{p'(my - nx) - (y - nz)}{q'(my - nx) + (x - mz)},$$

which reduces to

$$p'(x - mz) + q'(y - nz) = 0,$$

or when the conoid is right simply to

$$p'x + q'y = 0,$$

because then $m = 0$, $n = 0$.

(183.) The same results may be at once obtained from the consideration of the tangent plane; for $(x', y', z',)$ being any point on the surface, the equation of the tangent plane is

$$z - z' = p'(x - x') + q'(y - y'),$$

which touches the surface along the generatrix through (x', y', z'), and this being every where at the same distance z' from the horizontal plane, it follows that if in the above equation we put $z = z'$ the result

$$p'(x - x') + q'(y - y') = 0$$

will express the relation between the x, y of *every* point in this generatrix. But at that point where it cuts the straight directrix, the x, y have the relation

$$x = mz', \; y = nz',$$

so that, by substitution, we have

$$p'(mz' - x') + q'(nz' - y') = 0,$$

for the relation among the coordinates of every point $(x', y', z',)$ on the surface, which agrees with that deduced above.

Twisted Surfaces having Curvilinear Directrices only.

(184.) We now proceed to consider those surfaces which cannot

have a rectilinear directrix, or rather those whose directrices may be any lines whatever. We shall first suppose two directrices.

PROBLEM IV.

(185.) To determine the general equation of surfaces generated by a straight line which moves along any two directrices (D), (D′) whatever, and continues at the same time parallel to a fixed plane.

Taking as before the directing plane for that of xy, the equation of the generatrix in any position will be

$$z = \alpha,\ y = \beta x + \gamma \quad . \quad . \quad . \quad . \quad (1),$$

the parameters all varying with the varying positions of the generatrix. Let now the equations of the two fixed directrices be

$$(\mathrm{D}) \qquad \mathrm{F}\,(x, y, z) = 0, f\,(x, y, z) = 0 \quad . \quad . \quad . \quad . \quad (2)$$
$$(\mathrm{D}') \qquad \mathrm{F}_1\,(x, y, z) = 0, f_1\,(x, y, z) = 0 \quad . \quad . \quad . \quad . \quad (3).$$

Then the condition is, first that the generatrix meets (D), or that their equations (1), (2) exist together; hence by eliminating the coordinates of the common point from these four equations, we shall obviously obtain an equation containing only constants and the variable parameters α, β, γ, that is to say, we shall obtain among these parameters a relation

$$\Phi\,(\alpha, \beta, \gamma,) = 0.$$

Proceeding in the same manner with the equations (1), (3) which also exist together for a certain point, we obtain a second relation

$$\Psi\,(\alpha, \beta, \gamma,) = 0.$$

By means of these two equations we may eliminate any one of the parameters; therefore, eliminating first γ and then β, we have

$$\beta = \varphi{:}\alpha,\ \gamma = \psi{:}\alpha\,;$$

hence, substituting for these variable parameters their values in functions of the variable coordinates as furnished by equation (1), we have, for the general relation among these coordinates, the equation

$$xy = \varphi{:}z + \psi{:}z \quad . \quad . \quad . \quad . \quad (4).$$

This then is the general equation of all surfaces generated as announced, whatever be the form of the directrices; when these forms are given, the forms of φ and ψ become determinable by the above

process, and then the general equation (4) takes the particular form belonging to the individual surface.

(186.) Let us now determine the general equation of these surfaces in terms of the partial differential coefficients. Putting the equation (4) in the form

$$y - x\varphi{:}z = \psi{:}z,$$

the ratio of the partial coefficients of each side, taken relatively to x and y, will, by the principle in (58), be

$$\frac{-\varphi{:}z}{1} = \frac{p'}{q'}, \text{ that is, } \frac{p'}{q'} = -\varphi{:}z,$$

an equation from which the arbitrary function $\psi{:}z$ is eliminated. Applying the same principle to this last equation we have

$$\frac{d\left(\frac{p'}{q'}\right)}{dx} \div \frac{d\left(\frac{p'}{q'}\right)}{dy} = \frac{p'}{q'}$$

that is, putting according to the usual notation

$$\frac{dp'}{dx} = r', \frac{dq'}{dx} = \frac{dp'}{dy} = s', \frac{dq'}{dy} = t'$$

$$\frac{q'r' - p's'}{q'^2} \div \frac{q's' - p't'}{q'^2} = \frac{p'}{q'},$$

whence

$$q'^2r' - 2p'q's' + p'^2t' = 0,$$

an equation from which both the arbitrary functions are eliminated, and which must be fulfilled for every point in every surface generated as in the problem, whatever be the directrices. We see that as two arbitrary functions were to be eliminated, the process led to a partial differential equation of the second order.

PROBLEM V.

(187.) To determine the general equation of surfaces generated by the motion of a straight line along three curvilinear directrices (D) (D′), (D″).

We shall first remark that the motion of the generatrix is entirely governed by these conditions, for if we take any point on the first directrix (D) and conceive two cones whose bases are (D′), (D″) to

have this point for their common vertex, these cones will obviously intersect each other in all the straight lines that can be drawn from the point to the curves (D′), (D″), the positions of these lines are therefore fixed by these intersecting cones, and these are fixed by their bases; hence, all the lines that can be drawn from the point to the lines (D′), (D″) are determinate both in number and position, this being true for every point in (D), it follows that the surface generated by all these lines is determinate, and it is now required to find its equation.

As there is here no directing plane the equations of the generatrix in any position will take the form

$$x = \alpha z + \gamma, \; y = \beta z + \delta \; . \; . \; . \; . \; (1),$$

and, since it always has a point in common with (D), we may eliminate by means of the equation of (D) combined with these, the coordinates of that point: the result will furnish a condition among the variable parameters. In like manner, employing the equation of (D′) we shall arrive at another equation of condition, and, lastly, the equation of (D″) will furnish a third equation. By means of these three equations any two of the parameters α, β, γ, δ, may be eliminated, and we shall obtain three equations of the form

$$\beta = \varphi{:}\alpha, \; \gamma = \psi{:}\alpha, \; \delta = \pi{:}\alpha.$$

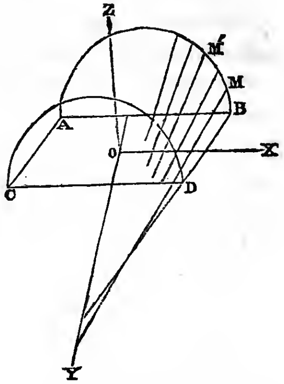
substituting these expressions for β, γ, δ, in the equations (1) we have

$$x = \alpha z + \psi{:}\alpha, \; y = z\varphi{:}\alpha + \pi{:}\alpha,$$

two equations which have place for every surface generated as proposed, the functions which fix the directrices being quite arbitrary. If these functions are known, or the directrices fixed, we may then eliminate the parameter α by means of these equations, and thus deduce the equation of the individual surface, but the general relations among the coordinates for all the surfaces of this family can be exhibited only by means of two equations as above. The general relation among the partial differential coefficients belonging to all this family of surfaces may, however, be ascertained in a single equation by eliminating, as in last problem, all the arbitrary functions by successive differentiation; this will lead to a partial differential equation of the third order, for which see *Monge's Application de l'Analyse à la Géométrie*, p. 195.

(188.) We shall terminate the present chapter with the following example:

On the opposite sides of the horizontal parallelogram AB DC are described two vertical semicircles, and perpendicular to their planes is drawn the straight line OY through the centre of the parallelogram; taking this straight line and the two semicircles as directrices, it is required to find the equation of the surface generated by a straight line moving along them.



Let the axes of coordinates be the perpendicular horizontal lines OX, OY, and the vertical OZ, then the equations of the three directrices will be

$$z = 0, z = 0 \quad . \quad . \quad . \quad . \quad (1)$$

$$y = -b, (x - a)^2 + z^2 = r^2 \quad . \quad . \quad . \quad . \quad (2)$$

$$y = +b, (x + a)^2 + z^2 = r^2 \quad . \quad . \quad . \quad . \quad (3).$$

The equations of the generatrix, since it always passes through a point (β, 0, 0) in the axis of y, will take the forms

$$x = \alpha\,(y - \beta), z = \gamma\,(y - \beta) \quad . \quad . \quad . \quad . \quad (4),$$

and the condition to be fulfilled by this line is, that it rests on each of the semicircles; or that at certain points, x, y, z, are the same in the equations (2), (4) and (3), (4); hence, eliminating these first from (2), (4), and then from (3), (4), we have these relations among the variable parameters, viz.

$$\{\alpha\,(b + \beta) + a\}^2 + \gamma^2\,(b + \beta)^2 = r^2 \quad . \quad . \quad . \quad . \quad (5)$$

$$\{\alpha\,(b + \beta) + a\}^2 + \gamma^2\,(b - \beta)^2 = r^2 \quad . \quad . \quad . \quad . \quad (6),$$

which, by subtraction, give

$$\beta\,(b\alpha^2 + a\alpha + b\beta^2) = 0.$$

This condition is satisfied by the value $\beta = 0$, but this is not admissible, since it would restrict the generatrix to pass always through the origin, and have no motion along OY; hence, dividing by $\frac{\beta}{b}$,

we have the relation

$$\alpha^2 + \gamma^2 + \frac{a\alpha}{b} = 0 \quad . \ . \ . \ . \ (7),$$

between the parameters α, γ.

Substituting the value of γ^2, given by this equation in (5), it becomes

$$(b^2 - \beta^2)\, a\alpha = b\,(r^2 - a^2) \quad . \ . \ . \ . \ (8),$$

and by means of these equations, together with those of the generatrix, we may readily eliminate the parameters; thus the values of α and γ, given by (4), are

$$\alpha = \frac{x}{y - \beta}, \ \gamma = \frac{z}{y - \beta},$$

and these, substituted in (7), give

$$\beta = y + \frac{b(x^2 + z^2)}{ax} \ \therefore \ \alpha = -\frac{ax^2}{b(x^2 + z^2)},$$

and finally, these substituted in (8) give for the surface the equation

$$\{y + \frac{b\,(x^2 + z^2)}{ax}\}^2 - b^2 = b^3\,(r^2 - a^2)\,(\frac{x^2 + z^2}{a^2x^2}),$$

which is the same as

$$\{axy + b\,(x^2 + z^2)\}^2 = b^3\,r^2\,x^2 + b^2\,(r^2 - a^2)\,z^2.$$

CHAPTER V.

ON DEVELOPABLE SURFACES AND ENVELOPES.

(189.) WHEN in an equation between three variables

$$\mathrm{F}\,(x, y, z, \alpha) = 0,$$

there enters an arbitrary constant α, that equation, by giving different values to α, will represent so many different surfaces all belonging to the same family. If we fix one of these by any determinate value of α, another, intersecting this, will be represented by changing α into $\alpha + h$, h being some finite value. If h be now continually diminished, the intersection will continually vary, and will become fixed

only when the varying surface becomes coincident with the fixed surface. In this position the intersection is said to belong to consecutive surfaces, and it may be determined both in form and position by a process similar to that employed at (105). Thus α being the only variable concerned in the intersections, let $u = \mathrm{F}(x, y, z, \alpha)$, now if α increase by h, $u' = \mathrm{F}(x, y, z, \alpha + h)$ which developed by Taylor's theorem, gives

$$u' = u + \frac{du}{d\alpha} h + \frac{d^2u}{d\alpha^2} \cdot \frac{h^2}{2} + \&c. = 0$$

but, since $u = 0$, therefore

$$\frac{du}{d\alpha} + \frac{d^2u}{d\alpha^2} \cdot \frac{h}{2} + \&c. = 0;$$

hence, when the surfaces are consecutive, that is, when $h = 0$, we have the following equations for determining the curve of intersection, viz.

$$\left.\begin{array}{l} du = 0 \\ \dfrac{du'}{d\alpha} = 0 \end{array}\right\} \ldots . (1),$$

these, therefore, are the equations of the curve, which is the intersection of the surface (1) with its consecutive surface.

If from these two equations we eliminate α, the result will be the general relation among the coordinates of every point in every such consecutive intersection throughout the whole family of surfaces, this resulting equation will therefore represent the surface which is the locus of all these consecutive intersections. This locus, moreover, touches each of the variable surfaces throughout their intersections; for differentiating the equation $\mathrm{F} = 0$ of any one of the variable surfaces, α being constant, we have

$$\frac{du}{dx} + \frac{du}{dz} p' = 0, \frac{du}{dy} + \frac{du}{dz} q' = 0,$$

and, differentiating the equation $\mathrm{F} = 0$ of the locus, α being variable, we have

$$\frac{du}{dx} + \frac{du}{dz} p' + \frac{du}{d\alpha} \cdot \frac{d\alpha}{dx} = 0, \frac{du}{dy} + \frac{du}{dz} q' + \frac{du}{d\alpha} \cdot \frac{d\alpha}{dy} = 0,$$

which equations are identical to those above, since

$$\frac{du}{d\alpha} = 0,$$

and therefore each pair give the same values for p' and q' consequently, at the points common to both surfaces they have common tangent planes. Hence the locus of the consecutive intersections touches, and envelopes all the variable surfaces; it is, therefore, called by Monge the *Envelope* of these surfaces.

(190.) If the envelope be formed by the consecutive intersections of *planes*, then, since from what has been just proved, the envelope is touched throughout each of the intersections by the corresponding plane; this envelope is such that the tangent plane at any point touches it throughout, the rectilinear generatrix passing through that point; and this is the characteristic property of a *developable surface*: hence a developable surface may be considered as the envelope of a family of planes represented by the general equation

$$z = \mathrm{A}x + \mathrm{B}y + \mathrm{D} \quad . \quad . \quad . \quad . \quad (1),$$

in which there enters a variable parameter α.

Now, to introduce this variable parameter in the most general manner possible into the equation (1), we ought to consider each of the coefficients A, B, C, to be functions of it, so that the general form will be

$$z = f\alpha + x\varphi\alpha + y\varphi\alpha,^{*}$$

and therefore the line of contact (189) or generatrix of the surface will be represented by the equations

$$\left.\begin{aligned} z &= f\alpha + x\varphi\alpha + y\psi\alpha \\ 0 &= f'\alpha + x\varphi'\alpha + y\psi'\alpha \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2).$$

When the forms of the functions f, φ, ψ, are fixed, the variable parameter α may be eliminated, and the resulting equation in x, y, z, will be that of the individual surface to which these particular forms belong. The equations (2), therefore, may be considered as represent-

* *Monge* says the general equation may always be put under the form

$$z = x\phi a + y\psi a + a,$$

which, however, seems to be incorrect, since it excludes those of the family comprehended in the equation†

$$z = x\phi a + y\psi a + c,$$

and which evidently generate conical surfaces, whose vertices are all on the axis of z, at the distance c, from the origin. The form in the text includes this class of equations, for fa may be constant without causing ϕa or ψa to become so.

† Monge in this observation is *not* incorrect, since a is indeterminate, one of its values will necessarily be equal to c. *Ed.*

ing the whole family of these surfaces, α in the first being a function of x and y, implied in the second.

(191.) 1. As an example, suppose it were required to determine the developable surface generated by the intersection of normal planes at every point in a curve of double curvature.

Representing the proposed curve by the equations

$$y' = \mathrm{F}x', z' = fx' \quad . \ . \ . \ . \ (1),$$

the general equation of the normal plane will be (129)

$$x - x' + \frac{dy'}{dx'}(y - y') + \frac{dz'}{dx'}(z - z') = 0 \quad . \ . \ . \ . \ (2),$$

in which the only variable parameter is x'; y' and z' being determinate functions of it given by the equations of the curve. Hence, differentiating with respect to x', we have

$$-(1 + \frac{dy'^2}{dx'^2} + \frac{dz'^2}{dx'^2}) + \frac{d^2y'}{dx'^2}(y - y') + \frac{d^2z'}{dx'^2}(z - z') = 0 \ . \ . \ . \ (3).$$

Now the functions of x', which enter the equations (2), (3), being given by (1), we may eliminate this parameter from them, and the resulting equation in x, y, z, will be that of the developable surface required.

(192.) 2. As a second example, let it be required to determine the developable surface which touches and embraces two given curve surfaces.

If we suppose one of these surfaces to be a luminous surface enlightening the other, the surface which we seek will obviously embrace all the rays which proceed from the bright surface to the dark one, and the curve of contact on this latter, will separate the illuminated and dark parts.

Let the equations of the two given surfaces be

$$\mathrm{F}_1(x_1, y_1, z_1) = 0, \mathrm{F}_2(x_2, y_2, z_2) = 0 \quad . \ . \ . \ (1),$$

then the equations to tangent planes to each will have the form

$$z - z_1, = p_1(x - x_1) + q_1(y - y_1) \quad . \ . \ . \ (2)$$

and

$$z - z_2 = p_2(x - x_2) + q_2(y - y_2),$$

and for these planes to belong to both surfaces, their equations must be identical, that is, we must have the conditions

$$p_1 = p_2, q_1 = q_2 \quad . \ . \ . \ (3),$$

$$z_1 - p_1x_1 - q_1y_2 = z_2 - p_2x_2 - q_2y_2 \quad . \ . \ . \quad (4).$$

By means of the six equations marked, five of the coordinates may be determined in terms of the sixth, x; hence, if these functions of x, be now substituted for their values in the remaining equation, we shall obtain a result containing only the variable parameter x_1, and which will consequently represent the family of planes which generate the developable surface sought. Calling this result $P = 0$, the generatrix of the surface will be given by the equations

$$P = 0, \frac{dP}{dx_1} = 0,$$

from which, eliminating x_1, we have the equation of the surface sought; and this equation, combined with that of each surface separately, will give the two curves of contact.

PROBLEM.

(193.) To determine the differential equation of developable surfaces in general.

The general equation of the generating plane, arranged according to the variable coordinates x, y, z, of any point in it, is

$$z = p'x + q'y + z' - p'x' - q'y' \quad . \ . \ . \quad (1)$$

and this plane remains the same for every point in the generatrix, as well as for the point (x', y', z'), so that the quantities

$$p', q', z' - p'x' - q'y' \quad . \ . \ . \quad (2),$$

remain constant, although x', y', z', all vary, provided this variation is confined to the rectilinear generatrix, for which y is always a function of x, but not else; hence, the conditions which restrict the point x', y', z', to the generatrix on which it is first assumed, is, that the differential coefficients derived from (2), y being considered as a function of x, are all 0, and it is plain that if any two be 0, the third will be 0 also; hence, differentiating the two first, we have

$$r' + s'\frac{dy}{dx} = 0, \text{ and } s' + t'\frac{dy}{dx} = 0,$$

where $\frac{dy}{dx}$ fixes the position of the rectilinear directrix for which the expressions (2), remain constant. Eliminating, then $\frac{dy}{dx}$, we obtain the following equation, which must hold for every directrix, viz.

$$r't' - s'^2 = 0 \;.\;.\;.\; (3).$$

This, therefore, is the equation which the differential of the equation of every developable surface must accord with; or, in usual terms, it is the differential equation of developable surfaces in general.

(194.) We shall exhibit another method of obtaining the equation (3), from the general equations (2), art. (190). Differentiating the first of these, in which α is a function of x and y implied in the second, we have the two partial differential equations

$$p' = f'\alpha \frac{d\alpha}{dx} + \varphi\alpha + x\varphi'\alpha \frac{d\alpha}{dx} + y\psi'\alpha \frac{d\alpha}{dx}$$

$$q' = f'\alpha \frac{d\alpha}{dy} + x\varphi'\alpha \frac{d\alpha}{dy} + \psi'\alpha + y\psi'\alpha \frac{d\alpha}{dy}$$

but, in virtue of the second of the equations (2), these become

$$p' = \varphi\alpha,\; q' = \psi\alpha,$$

consequently, p' must be a function of q', and may therefore be represented by

$$p' = \pi q'.$$

Eliminating now the arbitrary function π by differentiation, as in (58), we have

$$\frac{r'}{s'} = \frac{s'}{t'} \therefore r't' - s'^2 = 0,$$

as before.

(195.) We have as yet considered only the simplest class of surfaces, whose intersections, with their consecutive surfaces, are given by the general equations (1), art. (189), viz. plane surfaces, the intersections being straight lines. It is obvious, however, that whatever be the surfaces, the intersections are still given by the equations (1), and the envelope of these surfaces, found by eliminating from them the arbitrary parameter α. This parameter, however, may enter the equation of any particular family of surfaces in an infinite variety of different forms and ways; it may enter into only one of its terms, or be combined with several; a simple power only of it may enter, or a complicated function, and still, entering only as a parameter, the general equation, under all these changes, will still preserve the same character, and represent but one family of surfaces. With the envelope, however, it will be different; this depends as well on

the arbitrary parameter, as on the variables which enter the general equation, since the value of this parameter must be found from one of the equations (1), in terms of x, y, z, and this value substituted in the other for the equation of the envelope. Nevertheless, since, as just observed, the individual surfaces represented by the two equations (1), for every particular value of the parameter in whatever form it may enter, is always of the same degree, it follows that each individual intersection, (1), will uniformly be a curve of the same order, and which will change its order only when the order of the surface changes. This curve of intersection or of contact, common to all the envelopes of the same family of surfaces, is called, by *Monge*, the *characteristic.*

Considering any of the characteristics (1) separately, we may inquire what are the points in which it is intersected by the consecutive characteristic; and the method of determining these intersections is analogous to that already explained in (105) and (189), that is, we must combine with the equations (1) of this curve their differentials taken relatively to α; hence, the consecutive intersections for any particular position of the characteristic, will be determined by the equations

$$u = 0$$
$$\frac{du}{d\alpha} = 0$$
$$\frac{d^2u}{d\alpha^2} = 0,$$

each of these separately represent a surface, any two together a line common to both, and all three the point or points common to their intersection, α being considered constant. By solving these three equations for x, y, and z, we shall obviously obtain known values for the coördinates of the points of intersection required, which of course are all situated on the envelope.

Now, if from the three equations above, we eliminate α, we shall have two equations in x, y, z, existing together, which, being the same for the intersections of *every* pair of consecutive characteristics must represent the locus of these intersections, and be situated on the envelope. It will therefore be a line which touches and encompasses all the characteristics, in the same manner as the envelope touches

and embraces all the enveloped surfaces. It must then form an edge of the envelope, or the line in which its sheets terminate, and it is therefore called, by *Monge*, the *edge of regression of the envelope.* In the developable surfaces, we have seen that the characteristic is a straight line, and the consecutive intersections of the characteristic, in every position, obviously form the edge which limits the locus of the characteristics, that is, the developable surface.

The consideration of envelopes, characteristics, and edges of regression, have been successfully employed by *Monge*, and succeeding writers, to remove several difficulties in the higher departments of the integral calculus, that do not appear to be otherwise clearly explicable; but it would be out of place here to more than to hint at the importance of these researches; to pursue them to their fullest extent the advanced student must have recourse to the profound work of *Monge*, before referred to, viz. *Application de l'Analyse à la Géométrie.* We shall conclude the present chapter, with one example on the determination of the envelope.

(196.) The centre of a sphere of given radius moves along a given plane curve, it is required to determine the surface which envelopes the sphere in every position.

Let the equation of the given curve, along which the centre moves, be

$$\beta = \varphi\alpha \quad . \; . \; . \; . \quad (1),$$

so that for every abscissa α of this curve, the ordinate corresponding will be $\varphi\alpha$; therefore, the variable coordinates of the centre of the sphere are α, $\varphi\alpha$; hence its equation, in any position, is

$$(x-\alpha)^2 + (y-\varphi\alpha)^2 + z^2 = r^2 \quad . \; . \; . \; . \quad (2),$$

hence the equations of the characteristic are,

$$\left.\begin{array}{r}(x-\alpha)^2 + (y-\varphi\alpha)^2 + z^2 = r^2 \\ x-\alpha + (y-\varphi\alpha)\,\varphi'\alpha = 0\end{array}\right\} \quad . \; . \; . \; . \quad (3).$$

The last equation is that of a plane, passing through the point (α, $\varphi\alpha$), or centre of the sphere; it is, moreover, perpendicular to the tangent to the curve (1) at this point, for the equation of this tangent is

$$(\beta' - \beta) = \varphi'\alpha\,(\varphi\alpha' - \varphi\alpha),$$

and that above is

$$y - \varphi\alpha = -\frac{1}{\varphi'\alpha}\,(x-\alpha),$$

so that whatever be the form of φ, the characteristic is always a great circle of the moveable sphere, of which the plane is normal to the curve. The species of the curve which is the characteristic, being, however, constant, as observed in art. (195,) however φ and α may vary, the species may be at once determined by assuming $\alpha = 0$, $\varphi\alpha = 0$, which reduces the equations of the characteristic to

$$\left.\begin{array}{c} x^2 + y^2 + z^2 = r^2 \\ x = 0 \end{array}\right\} = y^2 + z^2 = r^2,$$

which belongs to a circle; the species, therefore, is a curve of the second order.

To determine the equation of the envelope, we must eliminate α from (3), and the resulting equation in x, y, z, will belong to the envelope; thus, if the curve (1) be a circle of radius a, then

$$\varphi\alpha = \sqrt{a^2 - \alpha^2} \therefore \varphi'\alpha = \frac{-\alpha}{\sqrt{a^2 - \alpha^2}}$$

substituting these values in the equations (3), they become

$$(x - \alpha)^2 + (y - \sqrt{a^2 - \alpha^2})^2 + z^2 = r^2,$$

$$\alpha y = x \sqrt{a^2 - \alpha^2},$$

and determining, from this last equation, the expression for α, and substituting it in the preceding, we shall obtain, finally,

$$(a \pm \sqrt{x^2 + y^2})^2 = r^2 - z^2,$$

for the equation of the envelope.

CHAPTER VI.

ON CURVES OF DOUBLE CURVATURE.

(197.) In the preliminary chapter to the present section, we investigated the expressions for the tangent lines and normal planes to these curves; we shall now discuss their general theory. As, however, in the course of this discussion, we shall sometimes have occasion to employ the differential expression for the arc of a curve of double

curvature, we shall commence by seeking the form of this expression.

(198.) We know that the projecting surface of every curve of double curvature, is a cylindrical surface, (see *Anal. Geom.*) if, therefore, this cylindrical surface be developed, the curve will become plane, and its length will be unaltered, and the curvilinear base of the projecting cylinder, which we shall here suppose to be vertical, will become a straight line on the plane of xy; hence, for the plane curve referred to this straight line t, and the axis of z, we shall have (86) the expression

$$(ds)^2 = (dz)^2 + (dt)^2,$$

but t being itself in reality the arc of a plane curve, we have

$$(dt)^2 = (dx)^2 + (dy)^2,$$

hence, by substitution,

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2,$$

which is the differential expression required.

Osculation of Curves of Double Curvature.

(199.) Let

$$y = fx, \quad z = \mathrm{F}x \quad . \quad . \quad . \quad . \quad (1),$$

and

$$\mathrm{Y} = \psi x, \ \mathrm{Z} = \Psi x \quad , \quad . \quad . \quad . \quad (2),$$

be the equations of two curves of double curvature, or rather of the projections of these curves on the planes of xy, xz: then, if we consider the constants a, b, c, &c. which enter the first pair of equations as known, and the constants A, B, C, &c. belonging to the second pair as arbitrary, these latter may be determined so that the curve to which they belong may touch the proposed or fixed curve (1), in any given point, more intimately than any other curve of the family (2). For, giving to x any increment, h, we have, by Taylor's theorem,

$$y' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\,\frac{h^2}{1 \cdot 2} + \&c.$$

$$\mathrm{Y}' = \mathrm{Y} + \frac{d\mathrm{Y}}{dx}h + \frac{d^2\mathrm{Y}}{dx^2}\,\frac{h^2}{1 \cdot 2} + \&c.$$

and it has been shown, (87), that if the constants which enter the first

of the equations (2), be determined, all of them from the conditions

$$y = \mathrm{Y}, \frac{dy}{dx} = \frac{d\mathrm{Y}}{dx}, \frac{d^2y}{dx^2} = \frac{d^2\mathrm{Y}}{dx^2}, \text{ \&c.} \quad \ldots \quad (3),$$

the projection of the curve (2), on the plane of xy, shall touch, more intimately, the projection of (1) on that plane, than the projection of any other curve of the family (2).

In like manner, if the constants which enter the second of the equations (2), be all of them determined from the conditions

$$z = \mathrm{Z}, \frac{dz}{dx} = \frac{d\mathrm{Z}}{dx}, \frac{d^2z}{dx^2} = \frac{d^2\mathrm{Z}}{dx^2}, \text{ \&c.} \quad \ldots \quad (4),$$

the projection of the curve (2) on the plane of xz, will touch more intimately the projection of (1) on that plane, than the projection of any other curve of the family (2).

It is clear, therefore, that if all the constants in the equations (2), be determined conformably to the conditions (3), (4), the curve (2) will touch more intimately the curve (1) in space, than any other curve of the family (2), and that the contact will be the less intimate as the conditions (3), (4), satisfied by the arbitrary constants, are fewer.

The conditions for simple contact, or contact of the first order, are evidently

$$y = \mathrm{Y}, z = \mathrm{Z}, \frac{dy}{dx} = \frac{d\mathrm{Y}}{dx}, \frac{dz}{dx} = \frac{d\mathrm{Z}}{dx} \quad \ldots \quad (5),$$

and, for *contact of the second order*, we must have the two additional conditions

$$\frac{d^2y}{dx^2} = \frac{d^2\mathrm{Y}}{dx^2}, \frac{d^2z}{dx^2} = \frac{d^2\mathrm{Z}}{dx^2},$$

and so on.

(200.) From these principles, we may very easily deduce the equations of the tangent at any point of a curve of double curvature.

Thus the equations of any straight line in space, are

$$y = \mathrm{A}x + \mathrm{B}; \; z = \mathrm{A}'x + \mathrm{B}' \quad \ldots \quad (1),$$

and these correspond to the equations (2) above, and as four arbitrary constants enter, the conditions (5) may be fulfilled by them; thus, taking the two last conditions, we have, by accenting the variables of the curve,

$$\frac{dy'}{dx'} = \text{A}, \frac{dz'}{dx'} = \text{A}',$$

and, therefore, the two first require that

$$\text{B} = y' - \frac{dy'}{dx'} x', \ \text{B}' = z' - \frac{dz'}{dx'} x'$$

so that the equations (1) of the tangent, through any point (x', y', z'), are,

$$\left.\begin{aligned} y - y' &= \frac{dy'}{dx'}(x - x') \\ z - z' &= \frac{dz'}{dx'}(x - x') \end{aligned}\right\} \ldots . \ (2).$$

PROBLEM I.

(201.) To determine the osculating circle, at any point in a curve of double curvature.

In finding the osculating circle, at any point of a plane curve, we had of course, the plane of that curve given, but, in the present case, we have to determine both the plane of the circle, and its radius. Now let us suppose that r is the radius of the osculating circle, and α, β, γ, are the coordinates of its centre, then it is plain, that the circle will be a great circle of the sphere whose equation is

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2 \ldots . \ (1),$$

and since the plane of the circle passes through the point $(\alpha, \beta, \gamma,)$ its equation must be of the form

$$x - \alpha + m(y - \beta) + n(z - c) = 0 \ldots . \ (2).$$

These two equations, combined with those of the proposed curve, give the values of x, y, z, common to all, and therefore belong to the point where the circle (1), (2), meets the curve. We have, therefore, to differentiate the equations (1), (2), successively, and to consider, agreeably to the conditions (3), (4), art. (199), that the resulting differential coefficients belong as well to the proposed curve at the point, as to this circle. For contact of the first order we have

$$(x - \alpha) + (y - \beta)p' + (z - c)q' = 0 \ldots . \ (3),$$

$$1 + mp' + nq' = 0 \ldots . \ (4),$$

and for contact of the second order we have, in addition,

$$1 + p'^2 + q'^2 + p''(y - \beta) + q''(z - \gamma) = 0 \quad . \quad . \quad . \quad . \quad (5),$$
$$mp'' + nq'' = 0 \quad . \quad . \quad . \quad . \quad (6).$$

All these six conditions, therefore, must exist for the contact at the point $(x, y, z,)$ in the proposed curve to be of the second order; and as the equations (1), (2), of the touching curve, contain six disposable constants, viz. α, β, γ, r, m, n, all these conditions may be fulfilled, but no more; hence, the circle, determined agreeably to these conditions, will touch the proposed curve more intimately than any other, that is, it will be the *osculating circle*. From equations (4) and (6) we get

$$m = \frac{q''}{q'p'' - p'q''}, \quad n = -\frac{p''}{q'p'' - p'q''},$$

hence, equation (2) becomes

$$x - \alpha + \frac{q''}{q'p'' - p'q''}(y - \beta) - \frac{p''}{q'p'' - p'q''}(z - \gamma) = 0,$$

or

$$z - \gamma = \frac{q'p'' - p'q''}{p''}(x - \alpha) + \frac{q''}{p''}(y - \beta) \quad . \quad . \quad . \quad . \quad (7)$$

hence, the three conditions (2), (4), (6), determine the plane of the osculating circle, and which is called the *osculating plane*, through the proposed point $(x, y, z.)$ Equation (7) then represents this plane.

For the coordinates of the centre of the osculating circle we have, from equations (1), (2), (3),

$$x - \alpha = \frac{(np' - mq')r}{\mathrm{M}}, \quad y - \beta = -\frac{(n - q')r}{\mathrm{M}},$$

$$z - \gamma = \frac{(m - p')r}{\mathrm{M}},$$

where M is put for the expression

$$\sqrt{\{(np' - mq')^2 + (n - q')^2 + (m - p')^2\}}.$$

Substituting these values in (5) we have, for the radius of the osculating circle,

$$r = \frac{(1 + p'^2 + q'^2)\,\mathrm{M}}{(n - q')\,p'' - (m - p')\,q''}.$$

Hence, putting for m and n the values already deduced, and restoring the value of M, we have

$$r = -\frac{(1 + p'^2 + q'^2)^{\frac{3}{2}}}{\sqrt{\{p''^2 + q''^2 + (p'q'' - q'p'')^2\}}},$$

$$\alpha = x - \frac{(1 + p'^2 + q'^2)(p'p'' + q'q'')}{p''^2 + q''^2 + (p'q'' - q'p'')^2},$$

$$\beta = y + \frac{(1 + p'^2 + q'^2)\{p'' - p'(p'p'' + q'q'')\}}{p''^2 + q''^2 + (p'q'' - q'p'')^2},$$

$$\gamma = z + \frac{(1 + p'^2 + q'^2)\{q'' - q'(p'p'' + q'q'')\}}{p''^2 + q''^2 + (p'q'' - q'p'')^2}.$$

(202.) The expression for r may be rendered more general, by considering the independent variable as arbitrary; in which case we have (66),

$$p'' = \frac{(d^2y)(dx) - (d^2x)(dy)}{(dx)^3},\ q'' = \frac{(d^2z)(dx) - (d^2x)(dz)}{(dx)^3}.$$

Also (198)

$$\frac{(ds)^2}{(dx)^2} = 1 + p'^2 + q'^2,$$

hence, making these substitutions in the above expression, we have

$$r = \frac{(ds)^3}{\sqrt{\{\overline{(dx)(d^2y) - (dy)(d^2x)}|^2 + \overline{(dz)(d^2x) - (dx)(d^2z)}|^2 + \overline{(dy)(d^2z) - (dz)(d^2y)}|^2\}}}$$

(203.) If it were required to determine the circle having contact of the first order, merely with the proposed curve, only the conditions (1), (2), (3), (4), must be satisfied; the conditions (2), (4), determine the plane of this circle, that is the tangent plane, but as the condition (4) leaves one of the constants m, n, arbitrary, the tangent plane is not fixed, but may take an infinite variety of positions; but as it must necessarily pass through the linear tangent, which is fixed, it follows that a plane through this, and revolving round it, is a tangent plane in every position, in one of which it touches the curve with a contact of the second order, and thus becomes the osculating plane.

(204.) There is another method of determining the equation of the osculating plane, very generally employed by French authors; they consider a curve of double curvature to have, at every point, two *consecutive elements*, or infinitely small contiguous arcs in the same plane, but not more, the plane of these elements being the osculating

plane at the point. The process, then, is to assume the equation of a plane through the point

$$x - x' + m(y - y') + n(z - z') = 0 \dots (1),$$

and to subject it to the condition of passing also through the points

$$(x + dx', y' + dy', z' + dz'),$$

and

$$x' + 2dx' + d^2x', y' + 2dy' + d^2y', z' + 2dz' + d^2z'.$$

Such a process, the student will at once perceive to be exceedingly exceptionable; for besides the vague notion attached to the infinitely small consecutive arcs, the expressions $x + dx$, $y + dy$, and the like, mean no more in the language of the *differential* calculus, than x, y, &c., for dx, dy, &c. are not infinitely small, but absolutely 0, as we have all along been careful to impress on the mind of the student. The process is, however, susceptible of improvement thus: suppose the plane (1) passing through one point (x', y', z') of the curve passes also through a second point, of which the abscissa is $x' + \Delta x'$, where $\Delta x'$ means *the increment of* x, then substituting $x' + \Delta x'$ for x', the equation (1) becomes

$$x - x' + m(y - y') + n(z - z') - (\Delta x' + m\Delta y' + n\Delta z')$$
$$= 0 \;.\;.\; (2),$$

which, in virtue of (1), is the same as

$$\Delta x' + m\Delta y' + n\Delta z' = 0,$$

or

$$1 + m\frac{\Delta y'}{\Delta x'} + n\frac{\Delta z'}{\Delta x'} = 0.$$

Suppose now that these two points merge into one, that is, let $\Delta x' = 0$, then

$$1 + m\frac{dy'}{dx'} + n\frac{dz'}{dx'} = 0 \dots . (3),$$

hence the plane becomes determinable by the conditions (1), (3).

Again, let this plane pass through a third point, $x' + \Delta x'$, then substituting this for x' in both the equations (1), (3), they will furnish the additional condition

$$m\,\Delta\frac{dy'}{dx'} + n\,\Delta\frac{dz'}{dx'} = 0,$$

hence, dividing by $\Delta x'$, and supposing this third point to coincide with the former, that is, supposing $\Delta x' = 0$, we have the new condition

$$m \frac{d^2y'}{dx'^2} + n \frac{d^2z'}{dx'^2} = 0 \quad . \quad . \quad . \quad . \quad (4).$$

The equations (3) and (4), determine m and n, and thence the plane (1), which is such as to pass through but one point of the curve, and at the same time to be so placed that the most minute variation from this position will cause it to pass through *three* points of the curve.

(205.) By whatever process the osculating plane is determined, the radius of the osculating circle may be easily found from considerations different from those at (201). For, as the linear tangent to the curve, must also be tangent to the osculating circle, it follows that the centre of this circle must be on the normal plane, as well as on the osculating plane; it must, therefore, lie in the line of intersection of this normal plane, with its consecutive normal plane; hence, if this line be determined, the combination of its equation with that of the osculating plane, will give the point sought. Now (189) the line of intersection of consecutive normal planes is

$$\left.\begin{array}{l} x - x' + p'(y - y') + q'(z - z') = 0 \\ p''(y - y') + q''(z - z') - p'^2 - q'^2 - 1 = 0 \end{array}\right\}$$

therefore, the centre is to be determined by combining these equations with that of the osculating plane, viz.

$$z - z' = \frac{q'p'' - p'q''}{p''}(x - x') + \frac{q''}{p''}(y - y'),$$

being precisely the same equations as those employed before, for the same purpose. If the origin be at the point, and the tangent be the axis of x, then x', y', z', p', q', are each 0; therefore, the equations of the line of intersection are

$$x = 0, \; y = -\frac{q''}{p''}z + \frac{1}{p''},$$

and the equation of the osculating plane

$$p''z - q''y = 0;$$

this, therefore, is perpendicular to the line of intersection. (*Anal. Geom.*)

(206.) The expressions in (201) for the coordinates of the centre

of the osculating circle will become very simple by introducing the substitutions furnished by art. (202); the results of these substitutions will be

$$\alpha = x + r^2 \frac{d^2x}{ds^2},\ \beta = y + r^2 \frac{d^2y}{ds^2},\ \gamma = z + r^2 \frac{d^2z}{ds^2},$$

the independent variable being s. (*See Note* D.)

PROBLEM II.

(207.) To determine the centre and radius of spherical curvature at any point in a curve of double curvature.

We are here required to determine a sphere in contact with the proposed curve at a given point, such that a line on its surface in the direction of the proposed may in the vicinity of the point be closer to the curve than if any other sphere were employed. In the direction of the curve the z and the y of the sphere must be both functions of x, so that the equation of the sphere is resolvable into two, corresponding to the equations (2) art. (199), which two equations belong to the curve which osculates the proposed. The actual resolution of the equation into two is obviously unnecessary; it will be sufficient in that equation to consider x as the only independent variable.

The general equation of a sphere is

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2 = r^2 \ldots . (1),$$

and the particular sphere required will be that whose constants are determined from the following differential equations:

$$x - \alpha + p'(y - \beta) + q'(z - \gamma) = 0 \ldots . (2)$$

$$p''(y - \beta) + q''(z - \gamma) + 1 + p'^2 + q'^2 = 0 \ldots . (3)$$

$$p'''(y - \beta) + q'''(z - \gamma) + 3(p'p'' + q'q'') = 0 \ldots . (4).$$

These four equations fix the values of the parameters α, β, γ, r, and, therefore, determine both the position and magnitude of the osculating sphere. If the origin of coordinates be at the proposed point, and the linear tangent be taken for the axis of x, the determination becomes easy, for x, y, z, being each $= 0$, as also p', q', the foregoing equations (2), (3), (4), become

$$\alpha = 0,\ p''\beta + q''\gamma = 1,\ p'''\beta + q'''\gamma = 0,$$

$$\therefore \beta = \frac{q'''}{p''q''' - p'''q''},\ \gamma = \frac{p'''}{q''p''' - q'''p''};$$

hence, by substitution in (1),

$$r = \frac{\sqrt{p'''^2 + q'''^2}}{p''q''' - q''p'''}.$$

(208.) We already know that if to every point in a curve of double curvature normal planes be drawn, the intersections of these planes with the consecutive normal planes will be the characteristics of the developable surface which they generate, and the intersection of any characteristic with the consecutive characteristic will be a point in the edge of regression, corresponding to the given point on the proposed curve. Now equation (2) above being that of the normal plane, this point is determined by precisely the same equations (2), (3), (4), as determine the centre of spherical curvature, these points, therefore, are one and the same, as might be expected; hence the locus of the centres of spherical curvature forms the edge of regression of the developable surface generated by the intersections of the consecutive normals. If then by means of one of the equations of the proposed curve and the three equations of condition mentioned we eliminate x, y, z, and then perform the same elimination by means of the other equation of the curve and the same conditions, we shall obtain two resulting equations in α, β, γ, which will be the equations of the edge of regression.

PROBLEM III.

(209.) To determine the points of inflexion in a curve of double curvature.

Since a curve of double curvature as its name implies has curvature in two directions, if at any point its curvature in one direction changes from concave to convex the point is called a point of *simple inflexion.* But if at the same point there is also a like change of curvature in the other direction, the point is then said to be one of *double inflexion.* In other words, if but one projection of the tangent crosses the projected curve the point is one of simple inflexion, but if the tangent cross the curve in both projections then the point is one of double inflexion. As in plane curves the tangent line has contact one degree higher at a point of inflexion, so here the contact of the osculating plane is one degree higher. Hence, at such a point besides the conditions in (201) which fix the osculating plane,

we must at a point of simple inflexion have the additional condition arising from differentiating (6), viz.

$$mp''' + nq'' = 0.$$

Eliminating $\frac{m}{n}$ from this and equation (5) we have

$$p''q''' = q''p''',$$

which condition renders the expression for the radius of spherical curvature at the point infinite, as it ought.* Unless, therefore, this condition exist, the point cannot be one of inflexion; but the point for which the condition holds may be one of inflexion, yet to determine this the curve must be examined in the vicinity of the point.

As to points of double inflection, it is evident from what has been said (121) with respect to plane curves that such points must fulfil the conditions

$$p'' = 0 \text{ or } \infty\,,\ q'' = 0 \text{ or } \infty\,,$$

and these render the radius r of absolute curvature infinite or 0.

Evolutes of Curves of Double Curvature.

(210.) In speaking of the evolutes of plane curves we observed (103,) that the evolute of any plane curve was such that if a string

* The French mathematicians consider a point of simple inflexion to be that at which three *consecutive elements* of the curve lie in the same plane. In a recent publication from the university of Cambridge the author has attempted to deduce the above equation of condition, by viewing the point of inflexion after the manner of the French. He has however confounded the consecutive elements of a curve with what the same writers term *consecutive points*; moreover, after having established the conditions necessary for the plane

$$z = Ax + By + C,$$

passing through one point $(x, y, z,)$ in the curve, to pass also through two points consecutive to this, viz. the conditions

$$\frac{dz}{dx} = A + B\frac{dy}{dx},\ \frac{dz_{,}}{dx} = A + B\frac{dy_{,}}{dx}$$

where $y_{,}$, $z_{,}$, belong to one of the consecutive points, it is inferred that

$$\frac{d^2z}{dx^2} = B\frac{d^2y}{dx^2},\ \frac{d^2z_{,}}{dx^2} = B\frac{d^2y_{,}}{dx^2}$$

an inference which is quite unwarrantable, and which cannot exist unless the plane pass through *four* consecutive points instead of *three*.

were wrapped round it and continued in the direction of its tangent till it reached a point in the involute curve, the unwinding of this string would cause its extremity to describe the involute. But besides the *plane* evolute hitherto considered, there are numberless curves of double curvature round which the string might be wound and continued in the direction of a tangent till it reached the involute, which would equally, by unwinding, describe this involute; and generally every curve, whether plane or of double curvature, has an infinite number of evolutes, as we are about now to show.

(211.) If through the centre of a circle, and perpendicular to its plane, an indefinite straight line be drawn, and any point whatever be taken in this line, then it is obvious that this point will be equally distant from every point in the circumference of the circle, so that, if a line be drawn from it to the circumference, this line, in revolving round the perpendicular under the same angle, will describe the circumference. Such a point is called a *pole* of the circle, so that every circle has an infinite number of poles, the locus of which is determined when the places of any two are given.

(212.) Now, as respects curves of double curvature, we have seen that the centre of the circle of absolute curvature corresponding to any point is in the line where the normal at this point is intersected by its consecutive normal, the centre itself being that point in this line where it pierces the osculating plane, which (205) is the plane drawn through the tangent line perpendicular to this line of intersection, or characteristic; hence the characteristic corresponding to any point in the curve is the locus of the poles of curvature at that point, and the intersection of this characteristic, with the perpendicular to it from the corresponding point of the curve, is that particular pole which is the centre of absolute curvature, the perpendicular itself being the radius.

As the locus of the poles corresponding to any point is no other than the characteristic, the locus of all the poles corresponding to all the points of the curve must be the locus of all the characteristics, and therefore (190) *a developable surface.*

(213.) Suppose now through any point, P, of the curve a normal plane is drawn of indefinite extent, the characteristic or line of poles corresponding to the point will be in this plane; let, therefore, any straight line be drawn from P to intersect this line of poles in the point

Q, and be continued indefinitely. If this normal plane be conceived to move, so that, while P describes the proposed curve, the plane continues to be normal, the characteristic will undergo a corresponding motion, and will generate the developable surface corresponding to the curve described by P, and this motion of the characteristic will cause a corresponding motion of the point Q, not only in space, but along the arbitrary line from P, which has no motion in the moving plane. As, therefore, Q moves along the characteristic successive portions QQ′ of the line, PQ will apply themselves to the surface which the moveable characteristic generates, and there form a curve to which always the unapplied portion QP is a tangent. Now the normal plane being in every position tangent to the surface throughout the whole length of the characteristic, it is obvious that, in the above generation of this surface, nothing more in effect has been done than the bending of the original normal plane, supposed flexible, into a developable surface. If, therefore, we now perform the reverse operation, that is, if we unbend the normal plane, the point P will describe the curve of double curvature, and the curve QQ′ traced on the developable surface will become the straight line PQ; so that the curve of double curvature may be described by the unwinding of a string wrapped about the curve Q′Q, and continued in the direction QP of its tangent, till it reaches the point P in the proposed curve. It follows, therefore, that the curve Q′Q is an evolute of the curve of double curvature proposed, and, moreover, that, as the line PQ originally drawn was quite arbitrary, *the proposed curve has an infinite number of evolutes situated on the developable surface, which is the locus of the poles of the proposed;* hence the locus of the poles is the locus of the evolutes.

If the original line PQ be perpendicular to the corresponding line of poles or characteristic, then, since this characteristic moves in the moving plane while PQ remains fixed, PQ cannot continue to be perpendicular to the characteristic; but the radius of absolute curvature is always perpendicular to the characteristic, this radius therefore cannot continue to intersect the characteristic in the point Q, so that *the locus of the centres of absolute curvature is not one of the evolutes of the proposed curve.*

(214.) Should the curve which we have all along considered of double curvature be plane, then, indeed, since the characteristics are

all parallel, and perpendicular to the plane of the curve, the line PQ once perpendicular will be always perpendicular to the characteristic, so that then Q will coincide with the centre of curvature, PQ being no other than the radius of curvature, the locus of the centres being the plane evolute before considered. But when PQ is not drawn perpendicular to the original characteristic, but is inclined to it at an angle α, then it always preserves this inclination during the generation of the cylindrical surface which is the locus of the poles, therefore every curvilinear evolute of a plane curve is a *helix* described on the surface of the cylinder, which is the locus of the poles of the plane curve.

Every curve traced on the surface of a sphere, has, for the locus of its evolutes, a conical surface whose vertex is at the centre of the sphere; because the normal planes to the curve being also normal planes to the spheric surface, all pass through the centre.

(215.) From what has now been said, it is obvious that if from any point in a curve a line be drawn to touch the developable surface which is the locus of its poles, and its prolongation be wound about the surface without *twisting*,* it will trace one of the evolutes, and, as the string may be drawn to touch the surface in every possible direction, it follows that every developable line on the surface will be an evolute. If the curve be plane, the evolutes are all on the cylindrical surface whose base is the plane evolute.

As obviously a developable line is the shortest on the surface that can join its extremities, it follows that the shortest distance between two points of an evolute measured on the surface is the arc of that evolute between them.

PROBLEM IV.

(216.) Having given the equations of a curve of double curvature to determine those of any one of its evolutes.

All the evolutes of the curve being on the same developable sur-

* This is what I understand *Monge* to mean, when he says (*App. de l'Anal. de Géom.* p. 348,) "si l'on plie *librement* sur cette surface le prolongement de cette tangente." It seems not improper to call such lines placed on a developable surface *developable lines*, and those which form curves on the developed surface *twisted lines*. Of these two species of lines all the former are evolutes, but none of the latter are.

face, the equation of this surface must be common to them all, and we have already seen (194) how the equation of the surface is to be determined, so that it only remains to find for each evolute a particular equation which distinguishes it from all the others, and determines its course on the developable surface. In order to this let us consider that each evolute must be such that the prolongation of its tangent at any point always cuts the involute, or, which is the same thing, the tangent to the projection of the evolute at any point passes through the corresponding point in the projection of the evolute; therefore, considering the plane of xy as that of projection, we have, for the tangent at any point (x', y') in the projected evolute,

$$\mathrm{Y} - y' = \frac{dy'}{dx'}(\mathrm{X} - x'),$$

and, since the same line passes through a point $(x, y,)$ in the projected involute, its equation is also

$$\mathrm{Y} - y' = \frac{y' - y}{x' - x}(\mathrm{X} - x')$$

$$\therefore \frac{dy'}{dx'} = \frac{y' - y}{x' - x};$$

hence, combining this equation with that of the developable surface, determined agreeably to the process pointed out in article, 194, and eliminating x, y being a given function of x, we shall have two equations in x', y', z', of which one will contain partial differential coefficients of the first order, and which together will represent all the evolutes. To find that particular one which is fixed by any proposed condition, it will be necessary to discover, by the aid of the integral calculus, the primitive equation from which the differential equation mentioned is deducible; this primitive equation will involve an arbitrary constant, whose value may be fixed by the proposed condition, and thus the equations of the particular evolute will be determined.

We shall terminate this section by subjoining a few miscellaneous propositions.

CHAPTER VII.

MISCELLANEOUS PROPOSITIONS.

PROPOSITION I.

(217.) To prove that the locus of all the linear tangents at any point of a curve surface is necessarily a plane.

This property we have hitherto assumed; it may, however, be demonstrated as follows:

Let the equation of any curve surface be

$$z = f(x, y) \quad . \quad . \quad . \quad . \quad (1),$$

x and y being the independent variables.

Through any given point on this surface let any curve be traced, then, the projection of this curve on the plane of xy will be represented by

$$y = \varphi x \quad . \quad . \quad . \quad . \quad (2),$$

which will equally represent the projecting cylinder; hence the combination of the equations (1), (2), completely determines the curve, and its projection on the plane of xz may be found by eliminating y from these equations; the result of this elimination will be the equation

$$z = f(x, \varphi x) = \psi x \quad . \quad . \quad . \quad . \quad (3),$$

therefore, since the linear tangent in space is projected into tangents to these two curves (2), (3), its equations must be

$$\left.\begin{aligned} y - y' &= \frac{d\varphi x'}{dx'}(x - x') \\ z - z' &= \frac{d\psi x'}{dx'}(x - x') \end{aligned}\right\}$$

where x', y', z', are the coordinates of the proposed point on the surface. Now $\frac{d\psi x}{dx}$ is the total differential coefficient derived from the function $z = f(x, y)$, in which y is considered as a function of x given by the equation (2), that is

$$\frac{d\psi x}{dx} = \left\{\frac{dz}{dx}\right\} = p' + q' \frac{d\varphi x}{dx};$$

hence, by substitution, the equations of the tangent in space become

$$\left.\begin{aligned} y - y' &= \frac{d\varphi x'}{dx'}(x - x') \\ z - z' &= (p' + q' \frac{d\varphi x}{dx})(x - x') \end{aligned}\right\} \ldots \ldots (4).$$

Now, to obtain the locus of the tangents whatever be the curve through the point (x', y', z'), we must eliminate the function φx, on which alone the nature of the curve depends. Executing then this elimination by means of the equations (4) and there results for the required locus the equation

$$z - z' = p'(x - x') + q'(y - y'),$$

which is that of a plane.

PROPOSITION II.

(218.) Given the algebraic equation of a curve surface to determine whether or not the surface has a centre.

That point is called the centre which bisects all the chords drawn through it, so that if the equation of the surface is satisfied for any constant values x', y', z', it will equally be satisfied for the same values taken negatively, that is, for $-x'$, $-y'$, $-z'$, provided the origin of coordinates be placed at the centre, so that if no point exists for the origin of coordinates, in reference to which the equation

$$f(x, y, z,) = 0$$

of the surface remains the same whether the signs of the variables be assumed all $+$ or all $-$, then we may conclude also that no centre exists.

The mode of proceeding, therefore, is to assume the indeterminates $x_{,}$, $y_{,}$, $z_{,}$ for the coordinates of the unknown centre, and to transport the origin of the axes to that point by substituting in the equation of the surface $x + x_{,}$, $y + y_{,}$, $z + z_{,}$ for x, y, z. This done we may readily deduce equations of condition which will give the proper values of $x_{,}$, $y_{,}$, $z_{,}$, if a centre exists, or will show, by their incongruity, that the surface has no centre. Thus, suppose the equation of the surface is of an even degree, then we must equate to 0 the coefficients

of all the *odd* powers and combinations of x, y, z, since the terms into which these enter would change signs when the variables change signs: we obtain in this way the equations of condition. If the equation of the surface be of an odd degree, then we must equate to zero the coefficients of all the *even* powers and combinations of x, y, z; so that only odd powers and combinations may effectively enter the equation, for then whether the variables be all $+$ or all $-$ the function $f(x, y, z,)$ will still be 0.

Now the differential calculus furnishes us at once with the means of obtaining the several expressions which we must equate to zero without actually substituting $x + x_{,}$, $y + y_{,}$, $z + z_{,}$, for x, y, z, in the equation of the surface. For if we conceive these substitutions made in the function $f(x, y, z)$, we may consider the result as arising from $x_{,}$, $y_{,}$, $z_{,}$, taking the respective increments x, y, z, and we know that every such function may by Taylor's theorem be developed according to the powers and combinations of the increments, and that the several terms of the development consist each of the partial differential coefficients of the preceding term, the first being $f(x_{,}, y_{,}, z_{,})$. Hence, if the coefficients of the first powers of x, y, z, are to be respectively zero, then we have to equate to zero each of the partial coefficients derived from $u_{,} = f(x_{,}, y_{,}, z_{,}) = 0$, or, which is the same thing, from $u = f(x, y, z,) = 0$ the proposed equation; if the coefficients of the second powers and combinations of x, y, z, are to be rendered each 0, then we shall have to equate to zero each partial coefficient derived from again differentiating, and so on.

As an illustration of this, let the general equation of surfaces of the second order

$$\left.\begin{array}{l} Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy \\ \qquad + 2Cx + 2C'y + 2C''z + E \end{array}\right\} = 0 = u \ldots (1)$$

be proposed, then the degree of the equation being even, the coefficients of the odd powers of the variables in the equation arising from putting $x + x_{,}$, $y + y_{,}$, $z + z_{,}$, for x, y, z, are to be equated to 0, and as the equation is but of the second degree, these odd powers will be of the first; hence we have merely to equate the first partial differential cofficients to 0, that is

$$\left.\begin{aligned}\frac{du}{dx} &= \mathrm{A}x + \mathrm{B}'z + \mathrm{B}''y + \mathrm{C} = 0\\ \frac{du}{dy} &= \mathrm{A}'y + \mathrm{B}z + \mathrm{B}''x + \mathrm{C}' = 0\\ \frac{du}{dz} &= \mathrm{A}''z + \mathrm{B}'z + \mathrm{B}y + \mathrm{C}'' = 0\end{aligned}\right\} \quad \ldots\ldots (2).$$

The values of x, y, z, deduced from these equations are the coordinates $x_{,}$, $y_{,}$, $z_{,}$ of the centre. These values may be represented by

$$x_{,} = \frac{\mathrm{N}'}{\mathrm{D}},\ y_{,} = \frac{\mathrm{N}'}{\mathrm{D}},\ z_{,} = \frac{\mathrm{N}''}{\mathrm{D}},$$

where

$$\mathrm{D} = \mathrm{AB}^2 + \mathrm{A}'\mathrm{B}'^2 + \mathrm{A}''\mathrm{B}''^2 - \mathrm{AA}'\mathrm{A}'' - 2\mathrm{BB}'\mathrm{B}'',$$

so that the surface has a centre if D is not 0, but if $\mathrm{D} = 0$ and the numerators all finite, the surface has no centre, and, lastly, if $\mathrm{D} = 0$ and either of the numerators, also 0, then the surface has an infinite number of centres, and is, therefore, cylindrical.

The equations of condition (2) are the same as those at page —— of the Analytical Geometry.

PROPOSITION III.

(219.) To determine the equation of the diametral plane in a surface of the second order which will be conjugate to a given system of parallel chords.

Let the inclinations of the chords to the axes be α, β, γ, then the equations of any one will be

$$x = mz + p,\ y = nz + q \quad \ldots\ldots (1),$$

where

$$m = \frac{\cos.\ \alpha}{\cos.\ \gamma},\ n = \frac{\cos.\ \beta}{\cos.\ \gamma}.$$

For the points common to this line and the surface we must combine this equation with equation (1) last proposition, and we shall have a result of the form

$$\mathrm{R}z^2 + \mathrm{S}z + \mathrm{T} = 0 \quad \ldots\ldots (2),$$

which equation will furnish the two values of z corresponding to the two extremities of the diameter, and therefore half the sum of these values will be the z of the middle, that is,

$$z = -\frac{S}{2R} \therefore 2Rz = S + 0 \; . \; . \; . \; . \; (3),$$

which is obviously the differential coefficient derived from (2), or, which is the same thing, the total differential coefficient derived from (1) last proposition, in which x and y are functions of z given by the equations (1). This differential coefficient is, therefore,

$$\left\{\frac{du}{dz}\right\} = \frac{du}{dx}\frac{dx}{dz} + \frac{du}{dy}\frac{dy}{dz} + \frac{du}{dz} = 0$$

$$= m\frac{du}{dx} + n\frac{du}{dy} + \frac{du}{dz} = 0, \; . \; . \; . \; . \; (4),$$

where p and q, the only quantities which vary with the chord, are eliminated; hence, this last equation represents the locus of the middle points of the chords or the diametral surface, and it is obviously a plane.

By actually effecting the differentiations indicated in equation (4) upon the equation (1) last proposition, we have for the equation of the required diametral plane,

$$m\,(Ax + B'z + B''y + C) + n\,(A'y + Bz + B''x + C')$$
$$+ A''z + By + B'x + C'' = 0,$$

or

$$(Am + B' + B''n)\,x + (A'n + B + B''m)\,y +$$
$$(A'' + Bn + B'm)\,z + Cm + C'n + C'' = 0.$$

PROPOSITION IV.

(220.) A straight line moves so that three given points in it constantly rest on the same three rectangular planes; required the surface which is the locus of any other point in it.

Let the proposed planes be taken for those of the coordinates, and let the coordinates of the generating point be x, y, z, and the invariable distances of this point from the three points resting on the planes of yz, xz, and xy, X, Y, Z. The coordinates of these three points will be

In the plane of yz, 0, y', z'
xz, x'', 0, z''
xy, x''', y''' 0.

Then, since the parts of any straight line are proportional to their projections on any plane, each part having the same inclination to it, it follows that if we project successively each of the parts X, Y, Z, on the three coordinate planes, we shall have the relations

$$\left.\begin{array}{l} \dfrac{x}{\mathrm{X}} = \dfrac{x - x''}{\mathrm{Y}} = \dfrac{x - x'''}{\mathrm{Z}} \\ \dfrac{y - y'}{\mathrm{X}} = \dfrac{y}{\mathrm{Y}} = \dfrac{y - y'''}{\mathrm{Z}} \\ \dfrac{z - z'}{\mathrm{X}} = \dfrac{z - z''}{\mathrm{Y}} = \dfrac{z}{\mathrm{Z}} \end{array}\right\} \dots\dots (1).$$

But the part X of the moveable straight line comprised between the generating point $(x, y, z,)$ and the point $(0, y_{,}, z_{,},)$, resting on the plane of y, z, has for its length the expression

$$\mathrm{X}^2 = x^2 + (y - y')^2 + (z - z')^2,$$

or

$$1 = \frac{x^2}{\mathrm{X}^2} = \frac{(y - y')^2}{\mathrm{X}^2} + \frac{(z - z')}{\mathrm{X}^2} \dots\dots (2)$$

but from the equations (1)

$$\frac{y - y'}{\mathrm{X}} = \frac{y}{\mathrm{Y}}, \quad \frac{z - z'}{\mathrm{X}} = \frac{z}{\mathrm{Z}};$$

hence, by substitution, (2) becomes

$$\frac{x^2}{\mathrm{X}^2} + \frac{y^2}{\mathrm{Y}^2} + \frac{z^2}{\mathrm{Z}^2} = 1,$$

consequently, the surface generated is always of the second order. The surface would still be of the second order if the three *directing planes* were oblique instead of rectangular, as is shown by *M. Dupin*, in his *Developpements*, p. 342, whence the above solution is taken.

PROPOSITION V.

(221.) To determine the *line of greatest inclination* through any point on a curve surface.

The property which distinguishes the line of greatest inclination through any point is this, viz. that at every point of it the linear tangent makes with the horizon a greater angle than any other tangent to the surface drawn through the same point of the curve. Now, as all the linear tangents through any point are in the tangent plane to

the surface at that point, that one which is perpendicular to the trace of the tangent plane will necessarily be the shortest, and therefore approach nearest to the perpendicular, that is, it will form a greater angle with the horizon than any of the others. We have, therefore, to determine the curve to which the linear tangent at every point is always perpendicular to the horizontal trace of the tangent plane to the surface through the same point, or, which is the same thing, the projection of the linear tangent on the plane of xy must be perpendicular to the trace of the tangent plane.

Now the equation of the projection of the linear tangent at any point is

$$y - y' = \frac{dy'}{dx'}(x - x'),$$

and, by putting $z = 0$ in the equation of the tangent plane, we have for the trace in the plane of xy the equation

$$-z' = p'(x - x') + q'(y - y'),$$

and, since these two lines are to be always perpendicular to each other, we must have throughout the curve the general condition.

$$\frac{dy}{dx} = \frac{q'}{p'} \therefore p'\frac{dy}{dx} - q' = 0,$$

p' and q' being derived from the equation of the surface; so that the values of these being obtained in terms of x and y, and substituted in the equation just deduced, the result will be the general differential equation belonging to the projection of every curve of greatest inclination that can be drawn on the proposed surface. To determine that passing through a particular point, or subject to a particular condition, we must, by help of the integral calculus, determine the general primitive equation from which the above is deducible, this primitive will involve an arbitrary constant which may be fixed by the proposed condition, and thus the particular line be represented.

PROPOSITION VI.

(222.) The six edges of any irregular tetraedron or triangular pyramid are opposed two by two, and the nearest distance of two opposite edges is called *breadth;* so that the tetraedron has *three*

breadths and *four* heights. It is required to demonstrate that in every tetraedron the sum of the reciprocals of the squares of the *breadths* is equal to the sum of the reciprocals of the squares of the *heights.*

Let the vertex of the tetraedron be taken for the origin of the rectangular coordinates, and let also one of the faces coincide with the plane of xz, then the coordinates of the three corners of the base will be

$$0,\ 0,\ z',\ |\ x'',\ 0,\ z'',\ |\ x''',\ y''',\ z''',$$

and the equations of the three edges terminating in the vertex will be

$$\left.\begin{array}{l} x = 0 \\ y = 0 \end{array}\right| \quad \left.\begin{array}{l} x = \dfrac{x''}{z''}z \\ y = 0 \end{array}\right| \quad \begin{array}{l} x = \dfrac{x'''}{z'''}z, \\ y = \dfrac{y'''}{z'''}z. \end{array}$$

Now the perpendicular distance between each of these edges and the opposite edge of the base will evidently be equal to the perpendicular demitted from the origin on a plane drawn through the latter edge, and parallel to the former. Hence, denoting the three planes through the edges of the base by

$$\mathrm{A}x + \mathrm{B}y + \mathrm{C}z = 1 \mid \mathrm{E}x + \mathrm{F}y + \mathrm{G}z = 1 \mid \mathrm{I}x + \mathrm{K}y + \mathrm{L}z = 1,$$

they must be drawn so as to fulfil the conditions (*See Anal. Geom.*)

$$\left.\begin{array}{l} \mathrm{C}z' = 1 \\ \mathrm{A}x'' + \mathrm{C}z'' = 1 \\ \mathrm{A}x''' + \mathrm{B}y''' + \mathrm{C}z''' = 0 \end{array}\right| \left.\begin{array}{l} \mathrm{G}z' = 1 \\ \mathrm{E}x''' + \mathrm{F}y''' + \mathrm{G}z''' = 1 \\ \mathrm{E}x'' + \mathrm{G}z'' = 0 \end{array}\right| \begin{array}{l} \mathrm{I}x'' + \mathrm{L}z'' = 1 \\ \mathrm{I}x''' + \mathrm{K}y''' + \mathrm{L}z''' = 1 \\ \mathrm{L}z' = 0 \end{array}$$

These conditions fix the following values for A, B, C, &c., viz.

$$\mathrm{C} = \frac{1}{z'},\ \mathrm{A} = \frac{1}{x''} - \frac{z''}{x''z'},\ \mathrm{B} = \frac{x'''z''}{x''y'''z'} - \frac{x'''}{x''y'''} - \frac{z'''}{y'''z'};$$

$$\mathrm{G} = \frac{1}{z'},\ \mathrm{E} = -\frac{z''}{x''z'},\ \mathrm{F} = \frac{1}{y'''} + \frac{x'''z''}{x''y'''z'} - \frac{z'''}{y'''z'};$$

$$\mathrm{L} = 0,\ \mathrm{I} = \frac{1}{x''},\ \mathrm{K} = \frac{1}{y'''} - \frac{x'''}{x''y'''}.$$

Hence, calling the breadths B, B′, B″, we have (*Anal. Geom.*)

$$\frac{1}{\mathrm{B}^2} = \mathrm{A}^2 + \mathrm{B}^2 + \mathrm{C}^2 = \frac{(y'''z' - y'''z'')^2 + (x'''z'' - x'''z' - x''z''')^2 + (x''y''')^2}{(x''y'''z')^2}$$

$$\frac{1}{\mathrm{B}'^2} = \mathrm{E}^2 + \mathrm{F}^2 + \mathrm{G}^2 = \frac{(y'''z')^2 + (x''z' + x'''z'' - x''z''')^2 + (x''y''')^2}{(x''y'''z')^2}$$

$$\frac{1}{B''^2} = I^2 + K^2 + L^2 = \frac{(y'''z')^2 + (x''z' - x''z')^2}{(x''y'''z')^2}.$$

$$\text{Hence } \frac{1}{B^2} + \frac{1}{B'^2} + \frac{1}{B''^2} =$$

$$\left.\begin{array}{l}(z'-z'')^2 y'''^2 + \{(z''-z')x''' - x''z'''\}^2 + \\ \{(z'-z''')x'' + x'''z''\}^2 + (y''z')^2 + (x''-x''')^2 z'^2\end{array}\right\} \div (x''y'''z')^2 \ldots\ldots (1).$$

Again, the expressions for the heights or perpendiculars demitted from each of the points

$$(0, 0, 0)\,;\ (0, 0, z')\,;\ (x'', 0, z'')\,;\ (x''', y''', z'''),$$

upon the plane which passes through the other three are, severally, (*Anal. Geom.*)

$$H^2 = \frac{(x''y'''z')^2}{(z''-z')^2\, y'''^2 + \{(z'''-z')\,x'' + (z'-z'')\,x'''\}^2 + (y'''x'')^2}$$

$$H'^2 = \frac{(x''y'''z')^2}{(z''y''')^2 + (x''z''' - x'''z'')^2 + (x''y''')^2}$$

$$H''^2 = \frac{(x''y'''z')^2}{(y'''z')^2 + (x'''z')^2},\ H'''^2 = \frac{(x''y'''z')^2}{(x''z')^2}$$

$$\therefore \frac{1}{H^2} + \frac{1}{H'^2} + \frac{1}{H''^2} + \frac{1}{H'''^2} =$$

$$\left.\begin{array}{l}(z''-z')^2 y'''^2 + \{(z'''-z')x'' + (z'-z'')x'''\}^2 + \\ 2(x''y''')^2 + (y'''z'')^2 + (x''z''' - x'''z'')^2 + (y'''^2 + x'''^2 + x''^2)z'^2\end{array}\right\}$$

$$\div (x''y'''z')^2 \ldots (2),$$

which expression is the same as that before deduced, and thus the theorem is established by a process purely analytical. This remarkable property was discovered by *M. Brianchon*, and formed the subject of the prize question in the Ladies' Diary for 1830: a solution upon different principles may be seen in the Diary for 1831.

END OF THE DIFFERENTIAL CALCULUS.

NOTES.

NOTE (A), *page* 19.

THE expressions for the differentials of circular functions are all readily derivable, as in the text, from the differential expressions for the sine and cosine. We here propose to show how these latter may be obtained, independently of the considerations in art. (14).

By multiplying together the expressions

$$\cos. A + \sin. A \sqrt{-1}, \quad \cos. A_1 + \sin. A_1 \sqrt{-1},$$

the product becomes

$$\cos. A \cos. A_1 - \sin. A \sin. A_1,$$
$$= (\cos. A \sin. A_1 + \sin. A \cos. A_1) \sqrt{-1}.$$

But (*Lacroix's Trigonometry.*)

$$\left.\begin{array}{l} \cos. A \cos\ A_1 - \sin. A \sin. A_1 = \cos. (A + A_1) \\ \cos. A \sin.\ A_1 + \sin. A \cos. A_1 = \sin. (A + A_1) \end{array}\right\} \ldots (1),$$

hence the product is

$$\cos. (A + A_1) + \sin. (A + A_1) \sqrt{-1},$$
$$= \cos. A' + \sin. A' \sqrt{-1}.*$$

Consequently, the product of this last expression, and

$$\cos. A_2 + \sin. A_2 \sqrt{-1},$$

is

$$\cos. (A' + A_2) + \sin. (A' + A_2) \sqrt{-1},†$$
$$= \cos. A'' + \sin. A'' \sqrt{-1},$$

the product of this last, and

$$\cos. A_3 + \sin. A_3 \sqrt{-1},$$

is

$$\cos. A''' + \sin. A''' \sqrt{-1}.$$

* Writing A' for $A + A_1$. † Writing A'' for $A' + A_2$ &c. *Ed.*

Hence, generally,

$$(\cos. A + \sin. A \sqrt{-1})(\cos. A_1 + \sin. A_1 \sqrt{-1})(\cos. A_2 + \sin. A_2 \sqrt{-1}) \;\&c. =$$
$$\cos. (A + A_1 + A_2 + A_3 + \&c.) \sin. (A + A_1 + A_2 + A_3 + \&c.) \sqrt{-1}.$$

Supposing, now,

$$A = A_1 = A_2 = A_3 = \&c.$$

this equation becomes

$$(\cos. A + \sin. A \sqrt{-1})^n = \cos. nA \pm \sin. nA \sqrt{-1},$$

or, since the radical may be taken either + or —

$$(\cos. A \pm \sin. A \sqrt{-1})^n = \cos. nA \pm \sin. nA \sqrt{-1},$$

which is the formula of *Demoivre*, n being any whole number.

Put $a = \frac{n}{m} A$ then

$$(\cos. a \pm \sin. a \sqrt{-1})^m = \cos. ma \pm \sin. ma \sqrt{-1},$$
$$= \cos. nA \pm \sin. nA \sqrt{-1} = (\cos. A \pm \sin. A \sqrt{-1})^n,$$

therefore, extracting the mth root,

$$\cos. \frac{n}{m} A \pm \sin. \frac{n}{m} A \sqrt{-1} = (\cos. A \pm \sin. A \sqrt{-1})^{\frac{n}{m}}$$

which is the formula when the exponent is fractional.

Having thus got *Demoivre's* formula, we may immediately deduce from it, as in art. (22), the series

$$\cos. nA = \cos.^n A - \frac{n(n-1)}{2} \cos.^{n-2} A \sin.^2 A + \&c.$$

$$\sin. nA = n \cos.^{n-1} A \sin. A - \frac{n(n-1)(n-2)}{2 . 3} \cos.^{n-3} A \sin.^3 A + \&c.$$

Let $n = \frac{1}{0}$, $\sin. A = 0 = A \therefore nA = \frac{0}{0}$ = any finite quality x, hence, by these substitutions, the foregoing series become

$$\cos. x = 1 - \frac{x^2}{1 \cdot 2} + \frac{x^4}{1 \cdot 2 . 3 . 4} - \&c.$$

$$\sin. x = x - \frac{x^3}{1 . 2 . 3} + \frac{x^5}{1 . 2 . 3 . 4 . 5} - \&c.$$

consequently,

$$d \sin. x = (1 - \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3 . 4} - \&c.) dx = \cos. xdx,$$

$$d \cos. x = -(x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \&c.)dx = -\sin. x dx$$

I had intended to have given here another method of arriving at the differentials of the sine and cosine, and to which allusion is made at page 41, but, upon close examination, I find that the process I had then in view is liable to objection, and is therefore best omitted.

NOTE (B), *page* 91.

Demonstration of the Theorems of Laplace and Lagrange.

Let it be required to develop the function

$$u = \Psi z \text{ where } z = \mathrm{F}\,(y + xfz).$$

By differentiating the second of these equations, first relatively to x, and then relatively to y, we have

$$\frac{dz}{dx} = \mathrm{F}'\,(y + xfz)(fz + xf'z\frac{dz}{dx})^{*},$$

$$\frac{dz}{dy} = \mathrm{F}'\,(y + xfz)\,(1 + xf'z\frac{dz}{dy}).$$

Multiplying the first by $\frac{dz}{dy}$ and the second by $\frac{dz}{dx}$, and subtracting, there results

$$\frac{dz}{dx} - fz\frac{dz}{dy} = 0 \;\therefore\; \frac{dz}{dx} = fz\frac{dz}{dy} \;\ldots\ldots\; (1),$$

but since u or Ψz depends only on z, we shall have

$$\frac{du}{dx} = \Psi' z\frac{dz}{dx},\; \frac{du}{dy} = \Psi' z\frac{dz}{dy},$$

therefore, eliminating $\Psi' z$, we get

$$\frac{du}{dx}\frac{dz}{dy} - \frac{du}{dy}\frac{dz}{dx} = 0,$$

or putting for $\frac{dz}{dx}$ its value (1), and making for abridgment $fz = \mathrm{Z}$,

* At page 89 we put $f'z$ to represent the differential coefficient of fz relatively to x; here the same symbol denotes the coefficient relatively to z.

this last expression becomes divisible by $\frac{dz}{dy}$ and reduces to

$$\frac{du}{dx} = Z\frac{du}{dy} \quad \ldots . \quad (A),$$

so that we may always substitute for $\frac{du}{dx}$ the quantity $Z\frac{du}{dy}$.

If we differentiate the preceding equation relatively to x, we shall obtain

$$\frac{d^2u}{dx^2} = \frac{dZ\frac{du}{dy}}{dx} \quad \ldots . \quad (2),$$

but the expression $Z\frac{du}{dy}$ being no other than $fz.\Psi'z\frac{dz}{dy}$, that is to say a function of z multiplied by $\frac{dz}{dy}$, we may consider it as the differential coefficict of some new function of z, which we may represent by u_1, and we shall then have

$$\frac{du_1}{dy} = Z\frac{du}{dy}, \text{ and } \frac{dZ\frac{du}{dy}}{dx} = \frac{d^2u_1}{dxdy}$$

therefore (2), inverting the order of the differentiations in this last expression,

$$\frac{d^2u}{dx^2} = \frac{d^2u_1}{dydx} = \frac{d\frac{du_1}{dx}}{dy} \quad \ldots . \quad (3).$$

now it must be observed that the relation (A) exists, whatever be the function u; it therefore exists for the function u_1, hence

$$\frac{du_1}{dx} = Z\frac{du_1}{dy}.$$

Substituting then in (3) for $\frac{du_1}{dx}$ its value here exhibited, and afterwards for $\frac{du_1}{dy}$ its equal $Z\frac{du}{dy}$, there results

$$\frac{d^2u}{dx^2} = \frac{dZ\frac{du_1}{dy}}{dy} = \frac{dZ^2\frac{du}{dy}}{dy} \quad \ldots . \quad (B).$$

Differentiating this last equation relatively to x, we shall obtain

$$\frac{d^3u}{dx^3} = \frac{d^2Z^2\frac{du}{dy}}{dxdy},$$

and considering, as before, the function $Z^2\frac{du}{dy}$ to be the differential coefficient of some new function of z, viz. u_2, we shall have, by inverting the order of the differentiations,

$$\frac{du^2}{dy} = Z^2\frac{du}{dy} \text{ and } \frac{d^3u}{dx^3} = \frac{d^2\frac{du_2}{dx}}{dy^2} \quad \ldots\ldots (4),$$

but the equation (A) subsisting for every function u, must have place for the function u_2, hence

$$\frac{du_2}{dx} = Z\frac{du_2}{dy},$$

therefore (4),

$$\frac{du_2}{dx} = Z^3\frac{du}{dy}, \frac{d^3u}{dx^3} = \frac{d^2Z^3\frac{du}{dy}}{dy^2} \quad \ldots\ldots (C).$$

The analogy among the expressions (A), (B), (C), is obvious, and we shall now show that this analogy continues uninterrupted; that is, generally, if

$$\frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-2}Z^{n-1}\frac{du}{dy}}{dy^{n-2}} \quad \ldots\ldots (M),$$

then

$$\frac{d^nu}{dx^n} = \frac{d^{n-1}Z^n\frac{du}{dy}}{dy^{n-1}} \quad \ldots\ldots (N).$$

For considering, as before, the function $Z^{n-1}\frac{du}{dy}$ to be the differentia coefficient of some new function of z, viz. u_{n-1}, so that

$$Z^{n-1}\frac{du}{dy} = \frac{du_{n-1}}{dy} \quad \ldots\ldots (5),$$

we have, by differentiating (M),

$$\frac{d^n u}{dx^n} = \frac{d^{n-1} \cdot \frac{du_{n-1}}{dx}}{dy^{n-1}},$$

but the equation (A) subsisting for every function of z subsists for u_{n-1} therefore

$$\frac{du_{n-1}}{dx} = Z \frac{du_{n-1}}{dy};$$

hence (5)

$$\frac{d^n u}{dx^n} = \frac{d^{n-1} Z^n \frac{du}{dy}}{dy^{n-1}} = \frac{d^{n-1} (fz) \frac{du}{dy}}{dy^{n-1}}.$$

Let now $x = 0$ in the original function, and in each of the coefficients $\frac{du}{dx}, \frac{d^2u}{dx^2} \ldots . \frac{d^n u}{dx^2}$, then we have

$$[u] = \Psi\text{:}Fy = \varphi y,\ [z] = Fy \therefore f[z] = f\text{:}Fy = \psi y$$

and

$$\left[\frac{d^n u}{dx^n}\right] = \frac{d^{n-1} (\psi y)^n \frac{d\varphi y}{dy}}{dy^{n-1}}.$$

Consequently, by Maclaurin's theorem,

$$u = \varphi y + \psi y \frac{d\varphi y}{dy} \cdot \frac{x}{1} + \frac{d \cdot (\psi y)^2 \frac{d\varphi y}{dy}}{dy} \cdot \frac{x^2}{1 \cdot 2} +$$
$$\frac{d^2 (\psi y)^3 \frac{d\varphi y}{dy}}{dy^2} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.$$

which is the theorem of *Laplace*.

The preceding investigation is taken with some slight variation from the large work of Lacroix, vol. i. p. 279.

In the particular case where

$$u = \Psi z \text{ and } z = y + xfz,$$

we have

$$[u] = \Psi y,\ [z] = Fy = y, f[z] = \psi y = fy:$$

hence the development is then

$$u = \Psi y + fy\frac{d\Psi y}{dy} \cdot \frac{x}{1} + \frac{d\,(fy)^2 \frac{d\Psi y}{dy}}{dy} \cdot \frac{x^2}{1 \cdot 2} + \&c.$$

which corresponds with the theorem of *Lagrange*, given at page 89.

Note (C), *page* 127.

Suppose the function $f(x + h)$ fails to be developable according to Taylor's series for the value $x = a$, and let the true development be represented by

$f(a + h) = fa + Ah^{\alpha} + \beta h^{\alpha+\beta} + Ch^{\alpha+\beta+\gamma} + \&c. \ldots . (1)$,

the terms being arranged according to the powers of h. It is required actually to find these terms.

Let the difference $f(a + h) - fa$ be divided by such a power of h, that the quotient will become neither 0 nor ∞; when $h = 0$, such a power of h can be no other than h^{α}, or that which ought to appear in the first term of this difference, for if the developed difference were divided by a lower power of h than this, the quotient would evidently be 0, when $h = 0$, and if it were divided by a higher power, the quotient would be infinite when $h = 0$; hence the proper divisor h^{α} being found, if we put $h = 0$ in the quotient, the result will be simply A; having thus found the true first term of the difference, let it be transposed to the other side, and we shall then have the difference

$$\frac{f(a + h) - fa}{h^{\alpha}} - A = Bh^{\beta} + Ch^{\beta+\gamma} + \&c.$$

Now the first side of this equation being known, we have, as before, to find that power of h, that it may be divided by, so that the quotient may be neither 0 nor infinite, when $h = 0$ this power will be h^{β}, and putting $h = 0$ in the quotient, the result is B, and in this manner it is plain that all the terms of the series are to be determined.

Let now $y = fx$ be the equation of a plane curve, and $Y = Fx$ the equation of another having a common point with the former, at which $x = a$, then

$$Fa = fa;$$

beyond this point the ordinates of the two curves for any abscissa $(a+h)$ will be $f(a+h)$ and $F(a+h)$, and their difference D will be

$$D = f(a+h) - F(a+h).$$

Let us now develop the function $F(a+h)$, as we have done the function $f(a+h)$, and we shall have

$$F(a+h) = Fa + A'h^{\alpha'} + B'h^{\alpha'+\beta'} + C'h^{\alpha'+\beta'+\gamma'} + \&c. \quad (2)$$

and it may be shown precisely, as at page 128, that the greater the number of leading terms in the two developments (1), (2), are the same, the nearer will the developments themselves approach to identity, so that no curve passing through the point common to the two former can approach so closely to either in the vicinity of that point, unless in the development of the ordinate the same number of leading terms agree with those in (1).

Lagrange observes (*Théorie des Fonctions Analytiques*, p. 184,) that we may call contact of the first order, contact of the second order, &c. the approximation of two curves, for which the two first terms, the three first terms, &c. are the same in the developments of the functions which represent the ordinates. All other authors who advert to this subject make the same remark, but it is erroneous, as a simple instance will show.

Let the developed ordinates be

$$A + Bh^2 + Ch^{\frac{9}{2}} + Dh^5 \quad + \&c.$$

$$A + Bh^2 + Ch^{\frac{9}{2}} + Mh^{\frac{11}{2}} \quad + \&c.$$

According to Lagrange the contact here would be of the second order, but by Taylor's theorem, these developments would be

$$A + 0h + Bh^2 + 0h^3 + 0h^4 + \&c.$$

$$A + 0h + Bh^2 + 0h^3 + 0h^4 + \&c.$$

and therefore, by the general principles established in Chap. II. Sect. II. the contact is of the fourth order, at least. If in the true developments the term next to $0h^4$ were the same in each, and the exponent of h a fraction between 4 and 5, while the term following differed in the two series, the contact might be properly said to be of the fifth order; but the sign of the difference of the two developments, when h is negative, will obviously depend on the fractional exponents of h, in the terms immediately beyond those which agree in the two series.

NOTE (D), *page* 224.

As the process from which the expressions for α, β, γ, given in article 203, are deduced, is not immediately obvious, we shall here exhibit it at length for the first of these expressions.

From the formulas for p'' and q'', at article 201, we immediately get for $p'p'' + q'q''$ the value

$$\frac{(d^2y)\,(dy)\,(dx) - (d^2x)\,(dy)^2 + (d^2z)\,(dz)\,(dx) - (d^2x)\,(dz)^2}{(dx)^4}$$

$$= \frac{(dx)\,\{(d^2y)\,(dy) + (d^2z)\,(dz)\} - (d^2x)\,\{(dy)^2 + (dz)^2\}}{(dx)^4}$$

$$= \frac{(dx)\,\{(ds)\,(d^2s) - (d^2x)\,(dx)^2\}^* - (d^2x)\,\{(ds)^2 - (dx)^2\}}{(dx)^4}$$

$$= \frac{(dx)\,(ds)\,(d^2s) - (d^2x)\,(ds)^2}{(dx)^4} = \frac{(ds)\,\{(dx)\,(d^2s) - (d^2x)\,(ds)\}}{(dx)^4}$$

Therefore, putting, for brevity, $\frac{\mathrm{A}}{(dx)^6}$ instead of the denominator, in the expression for α in article 201, we have

$$\alpha = x + \frac{(ds)^6}{\mathrm{A}} \cdot \frac{(d^2x)\,(ds) - (d^2s)\,(dx)}{(ds)^3}$$

$$= x + r^2\frac{d^2x}{ds}, \text{ article 66,}$$

and by a similar process the expressions for β and γ are obtained.

The expressions $\alpha - x$, $\beta - y$, $\gamma - z$ are obviously the projections of the radius of curvature r on the axes of x, y, z. But, if we represent the inclinations of the radius to the axes by λ, μ, ν, the expressions for the projections will be

$$r \cos.\,\lambda,\ r \cos.\,\mu,\ r \cos.\,\nu,$$

so that (206) we have, for the angles of inclination, the values

$$\cos.\,\lambda = r\frac{d^2x}{ds^2},\ \cos.\,\mu = r\frac{d^2y}{ds^2},\ \cos.\,\nu = r\frac{d^2z}{ds^2}.$$

By employing these expressions *M. Cauchy* has arrived by rather a novel process at the theorem of *Meusnier*, given at p. 181.

* Because from

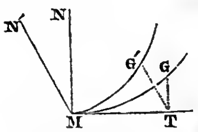
$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

we get

$$(ds)\,(d^2s) = (dx)\,(d^2x) + (dy)\,(d^2y) + (dz(\,(d^2z).$$

Let the equation of any curve surface be

$$u = \text{F}\,(x, y, z) = 0,$$



upon which is traced any curve MG′, determined by the equation

$$y = \varphi x,$$

joined to the preceding.

If through the tangent MT to this curve, and also through the normal MN of the surface, we draw a plane, we shall be furnished with a normal section MG, of which the radius of curvature r, at M, will be some portion of the normal MN. Also the radius of curvature r' of the assumed curve MG′, at the same point, will be some portion of the line MN′, perpendicular to the tangent MT.

Now, considering s to be the independent variable, we have, for the inclinations of r' to the axes the expressions above, viz.

$$r'\frac{d^2x}{ds^2},\ r'\frac{d^2y}{ds^2},\ r'\frac{d^2z}{ds^2},$$

and the inclinations of r or of MN to the axes are (127)

$$v\frac{du}{dx},\ v\frac{du}{dy},\ v\frac{du}{dz}.$$

Hence, calling the angle N′MN, between the two radii, ω, we have (*Anal. Geom.*)

$$\cos.\ \omega = v\,r'\left(\frac{du}{dx}\,\frac{d^2x}{ds^2} + \frac{du}{dy}\,\frac{d^2y}{ds^2} + \frac{du}{dz}\,\frac{d^2z}{ds^2}\right).$$

But the equation of the surface, considered as one of the equations of the curve MG′, gives after two successive differentiations, still regarding s as the independent variable

$$\frac{du}{dx}\,\frac{d^2x}{ds^2} + \frac{du}{dy}\,\frac{d^2y}{ds^2} + \frac{du}{dz}\,\frac{d^2z}{ds^2} =$$

$$\left\{\begin{array}{l} -\dfrac{d^2u}{dx^2}\,\dfrac{dx^2}{ds^2} - \dfrac{d^2u}{dy^2}\,\dfrac{dy^2}{ds^2} - \dfrac{d^2u}{dz^2}\,\dfrac{dz^2}{ds^2} \\ -2\dfrac{d^2u}{dx\,dy}\,\dfrac{dx\,dy}{ds^2} - 2\dfrac{d^2u}{dx\,dz}\,\dfrac{dx\,dz}{ds^2} - 2\dfrac{d^2u}{dy\,dz}\,\dfrac{dy\,dz}{ds^2}. \end{array}\right.$$

Now whatever be the curve MG′, provided only its tangent MT = x' remains unchanged, the second member of this last equation will remain unchanged, because the values of $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, which are the

same as those of $\frac{dx}{dx'}, \frac{dy}{dx'}, \frac{dz}{dx'}$, remain unchanged. Therefore this second member being substituted in the expression for cos. ω leads to a result of the form

$$r' = \mathrm{K} \cos. \omega,$$

K being a constant expression for all the curves on the proposed surface which touch MT, at the point M. Put now in this expression $\omega = 0$, then r' becomes r, therefore

$$r = \mathrm{K}, \text{ consequently } r' = r \cos. \omega,$$

which result comprehends the theorem of *Meusnier*, since, if the curve MG′ is plane, its plane will coincide with N′MT, and the angle ω of the two radii will become the angle formed by the plane N′MT of the oblique section with the plane NMT of the normal section passing through the same tangent MT.—*Leroy Analyse Appliquée à la Géométrie*, p. 268.

We may take this opportunity of remarking that, in our investigation of this theorem, at p. 182, it might easily have been shown, without referring to article 86, that

$$\frac{dx'^2}{dx^2} = 1 + \tan.^2 \theta,$$

because, by the right-angled triangle,

$$x'^2 = x^2 \sec.^2 \theta = x^2 (1 + \tan.^2 \theta)$$

$$\therefore \frac{dx'^2}{dx^2} = 1 + \tan.^2 \theta = \frac{ds^2}{dx^2}.$$

Note (E), *page* 106.

The erroneous doctrine adverted to at page 106 is laid down also by *Lacroix*, in his quarto treatise on the Calculus, vol. 1, p. 340, from whom, indeed, Mr. Jephson seems to have adopted it. The principle as stated by Lacroix is " que la série de Taylor devient illusoire pour toute valeur qui rend imaginaire l'un quelconque de ces terms; et que cela peut arriver sans que la fonction soit elle-même imaginaire." It is very remarkable that analysts should have hitherto held such imperfect notions respecting the failing cases of Taylor's theorem.

NOTES BY THE EDITOR.

NOTE (A′) *page* 15.

As the Algebra here referred to may not be in the hands of the student, we shall find the differential coefficient of a logarithmic function, by previously obtaining that of an exponential one, which is the course pursued by most writers on the calculus. Let

$$u = a^x \; . \; . \; . \; . \; (1),$$

in which if x be increased by h, we shall have

$$u' = a^{x+h} = a^x \times a^h.$$

Now in order to develop the last factor of this product, we suppose $a = 1 + b$, in order to subject it to the influence of the Binomial Theorem, we shall then have

$$a^h = (1+b)^h = 1 + hb + \frac{h}{1} \cdot \frac{h-1}{2} \cdot b^2 + \frac{h}{1} \cdot \frac{h-1}{2} \cdot \frac{h-2}{3} \cdot b^3 + \&c.$$

The multiplication indicated in the second number of this being executed, and the result ordered according to the powers of h, representing by sh^2 the sum of all the terms containing powers of h above the first, we shall then have

$$a^h = 1 + h\,(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \&c.) + sh^2$$

Both members of which being multiplied by a^x, designating the coefficient of h within the parentheses by c, we shall then have

$$a^x \times a^h = u' = (1 + ch + sh^2)\, a^x.$$

The primitive function being taken from this, leaves

$$u' - u = ca^x h + a^x sh^2,$$

whence

$$\frac{u' - u}{h} = ca^x + a^x sh,$$

which, when $h = 0$ becomes

$$\frac{du}{dx} = ca^x \quad . \; . \; . \; . \quad (2),$$

where

$$c = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \&c. \quad . \; . \; . \; . \quad (3).$$

It is thus perceived that the differential coefficient of an exponential function, is equal to that function multiplied by a constant number c, which is the above function of its base. We have from equation (2),

$$du = ca^x dx,$$

and we perceive from equation (1) that log. $u = x$, whence d log. $u = dx$; eliminating dx between this and the last, we have

$$du = ca^x d \log. u,$$

and

$$d \log. u = \frac{du}{ca^x} = \frac{1}{c} \cdot \frac{du}{u}.$$

The differential coefficient therefore of a logarithmic function is equal to the differential of the function divided by the function itself, multiplied by the constant $\frac{1}{c}$, the modulus of the system whose base is a. The modulus of the Naperian or Hyperbolic system of Logarithms being unity, we have

$$d \, . \, lu = \frac{du}{u},$$

lu representing the Naperian Log. of u.

Note (B′) *page* 21.

The leading part of article 15, in regard to the notation relative to inverse functions, though very plausible, is nevertheless calculated to mislead the student. For in the equation $x = F^{-1} y$, expressing the function that x is of y, the direct function being $y = Fx$, the symbols F and F^{-1} should not be considered as quantities or operated upon as such, since they here stand in place of the words *a function of*, the forms of both functions being different.

Note (C′) *page* 65.

Article 49 should have commenced with the equation

$$y = \mathrm{F}x,$$

and though the succeeding articles are full and ample on the subject, it may not be amiss to present the maxima and minima characteristics of functions in less technical language.

Remembering the note page 63,

$$\text{let } u = fx$$

be the proposed function to ascertain whether it admits of maxima or minima values; and if so, by what means they and the variable on which they depend may be discovered.

In the proposed function if the variable x first increase and then decrease by any quantity h, we shall then have

$$u = fx \; . \; . \; . \; . \; (1),$$

and by Taylor's Theorem,

$$u' = f(x+h) = u + \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^3}\frac{h^3}{1\,.\,2\,.\,3} + \frac{d^4u}{dx^4}\frac{h^4}{1\,.\,2\,.\,3\,.\,4} + \&\text{c.} \; . \; . \; . \; . \; . \; (2),$$

$$u'' = f(x-h) = u - \frac{du}{dx}\frac{h}{1} + \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} - \frac{d^3u}{dx^3}\frac{h^3}{1\,.\,2\,.\,3} + \frac{d^4u}{dx^4}\frac{h^4}{1\,.\,2\,.\,3\,.\,4} + \&\text{c.} \; . \; . \; . \; . \; . \; (3).$$

Now, in order that the function u under consideration may attain a maximum or minimum value (2) and (3), must be both less, or both greater than (1), and as h may be assumed so small that the term containing its first power may be greater than the sum of all the succeeding terms, (2) will be greater than u, while (3) will be less. Since the first differential coefficient has different signs in the two developments, the function therefore cannot attain maxima or minima values, unless this coefficient becomes zero. The roots of the equation $\frac{du}{dx} = 0$, will give such values of x as may render the function a maximum or a minimum; such values of the variable being

substituted in the second differential coefficient $\frac{d^2u}{dx^2}$ if these render its value any thing, we are certain the function may become a maximum if that value is negative, or a minimum if it be positive ; for in the first case (2) and (3) are both less than u, and in the second they are both greater. But if the same values of x render the second differential coefficient zero, as well as the first, we readily see that the third differential coefficient, must also become zero, in order that the function may admit of maxima or minima values : because this coefficient has different signs in (2) and (3), we then substitute the same values of x in the fourth differential coefficient, which has the same sign in (2) and (3), if these render it negative we shall have a maximum value of the function, and if positive a minimum value; but should this coefficient also vanish with the preceding ones, the next must be examined, and so on.

In order therefore to determine the values of x, which render the proposed function a maximum or minimum we must find the roots of the equation $\frac{du}{dx} = 0$, and substituted then in the succeeding differential coefficients, until we find one that does not vanish; if this be of an odd order, the roots we have employed will not render the function a maximum or minimum, but if it be of an even order, then if this coefficient be negative we have a maximum value of the function, but if positive a minimum value.

THE END.

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